ANALYSIS OF A SHELL OF TRANSLATION

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CERTIFICATE

Certified that this work on 'Analysis of a Shell of Translation' has been carried out under my supervision and that this has not been submitted elsewhere for a degree.

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ABSTRACT

This investigation concerns with the problem of a shell of translation, shallow in the longitudinal direction, acted upon by uniformly distributed vertical load and wind loads. A unit of a shell is formed by translating an inclined line vertically over a parabola. Such units can be combined to form an 'arched folded plate', the cross section of which is of a conventional folded plate where as the longitudinal axis is a parabolic curve. The transverse edges are either fixed or hinged. The longitudinal edges have been kept free, hinged or fixed.

The equilibrium equations, stress-strain laws and strain-displacement relations are derived according to linear theory of shells. The equilibrium equations are reduced to three equations in terms of fundamental displacements. Assuming the nature of displacements in the longitudinal direction, the unknown functional values in the transverse direction are evaluated by Runge-Kutta Gill integration procedure which yields the stresses and displacements in the structure. Several numerical results are presented.

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CHAPTER 1

INTRODUCTION

The economy of shell or folded plate structures is based on the use of main structural system as the building enclosure. A wide range of shells is made use of in Aeronautical, Civil and Naval structures which necessitates a high degree of sophistication in analysis. Most of the shells used for roofing comes under the category of thin shells where the ratio of deflection to thickness remains much smaller than unity.

The method of analysis of shells and plates under the 'thin shell theory' can be carried out on the assumptions proposed by Kirchoff |1|*; viz. (a) the straight fibres of the plates perpendicular to the middle surface before deformation remain so after deformation and do not change their length, (b) the normal stress acting on planes parallel to the middle surface can be neglected in comparison with the other stresses without loss of much accuracy. This was elucidated by Novozhilov |2| who proved that the errors introduced by Kirchoff's assumption are of the order of h/R compared with unity. The relation between forces, moments and

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* Numbers within || designate references at end of thesis.

displacements at the middle surface are accepted with Love's approximation which omits higher order terms in the elasticity relations for the sake of consistency.

Much work has been done in Soviet Union where the mathematical theory of shells is developed by Goldenveizer, Novozhilov, Galerkin and others. Vlazov has proposed many a simplification suitable for practical designs and such theory is generally acknowledged as 'engineering theory' of shells. Goldenveizer formulated the conditions of compatability of strains and showed the possibility of identically satisfying the equations of equillibrium by means of stress function, which was also in similar lines proposed by Lure |3|. Vlazov |4| expressed the internal forces and deformations of a shell by two scalar functions. The limit of applicability of Vlazov's shallow shell theory are partially elucidated in the works of Goldenveizer |5|, and Lure |3|.

For translational shells, various numerical integration procedures have gained great interest with the increasing use of electronic computers. Hedgren and Billington |6| combined the stress-displacement relations and equillibrium relations into eight first order

ordinary linear differential equations. Assuming a Levy type of solutions for displacements u, v, w, the unknowns are solved by Runge-Kutta integration procedure taking only one term in the respective series. A good accuracy of solution is claimed by taking only a single term. Bouma |7| and Apeland |8| have proceeded in the same lines where they reduce the equillibrium equation to eighth order partial differential equation in a stress-displacement function. All the unknown stresses and displacements are given in terms of this function. Mandel and Brennan |9| used finite difference technique to solve for the unknowns u, v, w at each section for a fixed edge hyperbolic paraboloid.

It should be mentioned here that a finite difference technique is direct but the accuracy of the solution depends, to an extent, upon the grid size which is limited by the core space of the computer used. Sometimes, to accomodate all finite difference equations, fictitious nodes had to be established beyond the shell boundaries. To match the number of unknown nodal quantities, some central difference equation had to be replaced by backward difference equations. The number of matrix operations to be done are also conducive for error propagation.

It is also relevent to see to the analysis of folded plates by the classical plate theory and elasticity relations. Goldberg and Leve |10| developed a solution for the stresses in such structures by combining the equations of the classical plate theory and the elasticity equations. Applied loading is approximated by a Fourier series expansion. Equations are derived for the case of a longitudinally simple span structure which relate each joint force to a linear combination of joint displacements. 4.n number of simultaneous algebraic equations are to be solved for each harmonic of Fourier expansion where 'n' is the number of joints with unknown force and displacements. In another paper by Goldberg, Glauz and Setlur |11|, the equations-for slabs simply suported at their ends - which govern the deformation and tractions are written as a system of eight first order differential equations. These, together with the joint transformation, are integrated using Runge-Kutta fourth order process readily giving the displacements, rotations and stresses at all points in the antinodal lines.

This thesis concerns with the analysis of translational shell which is formed by translating an inclined line over a parabola. The shell is assumed to be fixed or

hinged at the transverse supports and the longitudinal boundary may be free, hinged or fixed. The shell is analysed considering it to be shallow in the longitudinal direction on the assumption that the squares of the first order surface derivatives are negligible compared with unity. This is also asserted justifyable by E. Reissner |12| who states that the shallow shell theory will be accurate enough for practical purposes as long as the first order surface derivative is less than or equal to 1/2.

Basic shell equations are derived in chapter 2. The shell equations when shallow in longitudinal direction are given in chapter 3, which deals with the fundamental equilibrium equations, in non-dimensionalised form.

Chapter 4 deals with the displacement functions corresponding the transverse support boundary conditions. Longitudinal boundary conditions are also given in detail in this chapter.

The numerical analysis part of the problem is dealt with in chapter 5. The chapter 6 gives the conclusions drawn from the results.

CHAPTER 2

BASIC SHELL EQUATIONS

2.1 Geometry of the Shell:

The surface under study is obtained by translating an inclined line over a parabola. The equation of such a surface in the rectangular cartesian coordinates (x, y, z) is given by (see figure 1),

$$z = \left(- \frac{4H}{L^2} \right) x^2 - \tan\theta . y \qquad (2.1.1)$$

where H = height of the shell at $x = 0$

θ = inclination of the generating line to the y direction

L = span of the shell in longitudinal direction

The position vector to any point is given by,

$$\bar{R} = x\,\bar{i} + y\,\bar{j} + z\,\bar{k} \qquad (2.1.2)$$

where $\bar{i}$, $\bar{j}$, $\bar{k}$ are the unit vectors along x, y, z directions respectively.

α and β are the curvilinear coordinate lines and in the sequel subscripts 1 and 2 are used for α and β directions respectively. The vectors $\bar{e}_1$ and $\bar{e}_2$ are unit

tangent vectors along α and β directions and $\bar{e}_3$ is the unit normal vector,

$$\bar{e}_3 = \frac{\bar{e}_1 \times \bar{e}_2}{AB \sin \chi} \tag{2.1.3}$$

where A = Lame's parameter in α direction

B = Lame's parameter in β direction

χ = angle between $\bar{e}_1$ and $\bar{e}_2$

This nonorthogonal triad can be expressed as,

$$\begin{bmatrix} \bar{e}_1 \\ \bar{e}_2 \\ \bar{e}_3 \end{bmatrix} = \begin{bmatrix} \frac{1}{A} & 0 & \frac{z_x}{A} \\ 0 & \frac{1}{B} & \frac{z_y}{B} \\ \frac{-z_x}{D} & \frac{-z_y}{D} & \frac{1}{D} \end{bmatrix} \begin{bmatrix} \bar{i} \\ \bar{j} \\ \bar{k} \end{bmatrix} \tag{2.1.4}$$

where $z_x = \frac{\partial z}{\partial x}$, $z_y = \frac{\partial z}{\partial y}$

$A = (1 + z_x^2)^{1/2}$, $B = (1 + z_y^2)^{1/2}$,

$D = (1 + z_x^2 + z_y^2)^{1/2}$

The expression for the square of the differential of arc length, known as the first fundamental form of the surface, is given by

$$I = ds^2 = A^2 dx^2 + 2AB \cos\chi \; dxdy + B^2 dy^2$$

where $\cos \chi = \frac{z_x z_y}{AB}$

The second fundamental form for this surface is given as,

$$II = L dx^2 + 2M \, dxdy + N \, dy^2 \tag{2.1.5}$$

where $L = \frac{z_{xx}}{D}$, $M = -\frac{z_{yy}}{D} = 0$, $N = -\frac{2z_{xy}}{D} = 0$

also, $\frac{1}{R_1}$ = radius of curvature in α direction $= -\frac{L}{A^2}$

$\frac{1}{R_2}$ = radius of curvature in β direction $= -\frac{N}{B^2} = 0$

$\frac{1}{R_{12}}$ = cross-curvature $= \frac{M}{AB} = 0$

The gauss-weingarten equation are represented by,

$$\begin{bmatrix} \frac{\partial^2 \bar{R}}{\partial x^2} \\ \frac{\partial^2 \bar{R}}{\partial y^2} \\ \frac{\partial^2 \bar{R}}{\partial x \partial y} \end{bmatrix} = \begin{bmatrix} \Gamma_{11}^1 & \Gamma_{11}^2 & L \\ \Gamma_{12}^1 & \Gamma_{12}^2 & M \\ \Gamma_{22}^1 & \Gamma_{22}^2 & N \end{bmatrix} \begin{bmatrix} \frac{\partial \bar{R}}{\partial x} \\ \frac{\partial \bar{R}}{\partial y} \\ \bar{e}_3 \end{bmatrix} \tag{2.1.6}$$

Where the value of christoffel symbols are given as,

$$\Gamma_{11}^1 = \frac{z_x z_{xx}}{D^2}$$

$$\Gamma_{11}^2 = \frac{z_y z_{xx}}{D^2} \qquad \Gamma_{12}^1 = \Gamma_{12}^2 = \Gamma_{22}^1 = \Gamma_{22}^2 = 0$$

2.2 Vectors of Elastic Displacement and Elastic Rotation of the Middle Surface:

Assuming u, v, w are the displacements of a point on the middle surface of the shell in α, β and normal directions respectively (see figure 2), we can write a vector of elastic displacement $\bar{U}$ of the middle surface of the shell such that,

$$\bar{U} = u\,\bar{e}_1 + v\,\bar{e}_2 - w\,\bar{e}_3 \qquad (2.2.1)$$

Introducing an auxilliary trihedron such that (see figuge 3),

$$\bar{e}_1.\bar{e}_1' = \sin\chi = \bar{e}_2'.\bar{e}_2$$

$$\bar{e}_1'.\bar{e}_2 = 0 = \bar{e}_2'.\bar{e}_1$$

$$\bar{e}_1' \times \bar{e}_2' = \sin\chi\ \bar{e}_3$$

i.e.

$$\bar{e}_1' = \frac{\bar{e}_1}{\sin} - \cot\chi\ \bar{e}_2 \qquad (2.2.2)$$

$$\bar{e}_2' = \frac{\bar{e}_2}{\sin} - \cot\chi\ \bar{e}_1$$

The following notations stand for the strains and rotations of the surface,

e_1 = strain in the α direction

e_2 = strain in the β direction

ω_1 = angle through which the element directed along $\bar{e}_1$ rotates towards $\bar{e}_2$ in the tangent plane.

ω_2 = angle through which the element along $\bar{e}_2$ rotates towards $\bar{e}_1$ in the tangent plane.

γ_1 = angle through which the element directed along the vector $\bar{e}_1$ rotates towards vector $\bar{e}_3$ in the plane $(\bar{e}_1, \bar{e}_3)$.

γ_2 = angle through which the element directed along $\bar{e}_2$ towards $\bar{e}_3$ in plane $(\bar{e}_2, \bar{e}_3)$.

$\omega = \omega_1 + \omega_2$ = change in angle χ between the coordinate lines.

$\delta = \frac{\omega_2 - \omega_1}{2}$ = a measure of the angle of rotation of an element of the middle surface around $\bar{e}_3$ from $\bar{e}_2$ to $\bar{e}_1$.

It can be shown that |30|,

$$e_1 = \frac{\bar{U}_1}{A} \cdot \bar{e}_1' = \frac{1}{A}\frac{\partial u}{\partial x} + \frac{z_x z_y}{A^2 B}\ \frac{\partial v}{\partial x} - \frac{w}{R_1}$$

$$e_2 = \frac{\bar{U}_2}{B} \cdot \bar{e}_2 = \frac{z_x z_y}{AB^2}\frac{\partial u}{\partial y} + \frac{1}{B}\frac{\partial v}{\partial y}$$

$$\omega_1 = \frac{\bar{U}_1}{A} \cdot \bar{e}_2' = u\,\frac{z_y z_{xx}}{A^3 D} + \frac{D}{A^2 B}\frac{\partial v}{\partial x} + \frac{w}{R_1}\cot\chi$$

$$\omega_2 = \frac{\bar{U}_2}{B} \cdot \bar{e}_1' = \frac{D}{AB^2}\frac{\partial u}{\partial y}$$

where $\bar{U}_1 = \partial\bar{U}/\partial x, \quad \bar{U}_2 = \partial\bar{U}/\partial y$ (2.2.3)

We can consider a vector $\bar{\Omega}$ called the vector of elastic rotation such that it satisfies the relation below,

$$\bar{\Omega} \cdot \bar{e}_2' = \gamma_1$$

$$\bar{\Omega} \cdot \bar{e}_1' = -\gamma_2$$

$$\bar{\Omega} \cdot \bar{e}_3 = \delta$$

i.e. $$\bar{\Omega} = \frac{-\gamma_2}{\sin\chi}\,\bar{e}_1 + \frac{\gamma_1}{\sin\chi}\,\bar{e}_2 + \delta\,\bar{e}_3 \qquad (2.2.4)$$

Here the vectors $\bar{\Omega}$ and $\bar{U}$ are not independent and it can be shown that,

$$\gamma_1 = \frac{\bar{U}_1}{A} \cdot \bar{e}_3 = -\frac{u}{R_1} - \frac{1}{A}\frac{\partial w}{\partial x} \qquad (2.2.5)$$

$$\gamma_2 = \frac{\bar{U}_2}{B} \cdot \bar{e}_3 = -\frac{1}{B}\frac{\partial w}{\partial y}$$

and the components of bending deformation are given by,

$$k_1 = -\frac{\bar{\Omega}_1}{A} \cdot \bar{e}_2 = \frac{-\gamma_2}{A^3 D} z_y z_{xx} + \frac{z_x z_y}{A^2 B}\frac{\partial}{\partial x}\left(\frac{\gamma_2 AB}{D}\right)$$

$$- \frac{1}{A}\frac{\partial}{\partial x}\left(\frac{\gamma_1 AB}{D}\right)$$

$$k_2 = \frac{\bar{\Omega}_2}{B} \cdot \bar{e}_1 = -\frac{1}{B}\frac{\partial}{\partial y}\left(\frac{\gamma_2 AB}{D}\right)$$

$$+ \frac{z_x z_y}{AB^2}\frac{\partial}{\partial y}\left(\frac{\gamma_1 AB}{D}\right)$$

$$\tau^1 = \frac{\bar{\Omega}_1}{A} \cdot \bar{e}_1 = -\frac{1}{A}\frac{\partial}{\partial x}\left(\frac{\gamma_2 AB}{D}\right) + \frac{z_x z_y}{A^2 B}\frac{\partial}{\partial x}\left(\frac{\gamma_1 AB}{D}\right) + \frac{\delta}{R_1}$$

$$\tau^2 = \frac{\bar{\Omega}_2}{B} \cdot \bar{e}_2 = -\frac{z_x z_y}{BD}\frac{\partial}{\partial y}(\gamma_2) + \frac{A}{D}\frac{\partial \gamma_1}{\partial y} \qquad (2.2.6)$$

where $\bar{\Omega}_1 = \frac{\partial \bar{\Omega}}{\partial x}$; $\Omega_2 = \frac{\partial \bar{\Omega}}{\partial y}$

Introducing new quantities,

$$\psi_1 = \frac{\bar{\Omega}_1}{A} \cdot \bar{e}_3 = \frac{\gamma_2 AB}{DR_1} + \frac{1}{A}\frac{\partial}{\partial x}(\delta)$$

$$\psi_2 = \frac{\bar{\Omega}_2}{B} \cdot \bar{e}_3 = \frac{1}{B}\frac{\partial}{\partial Y}(\delta)$$

2.3 Stress Resultants and Stress Couple Resultants:

The force vector and moment vector along α - curve can be written as (see figure 4. and 5.)

$$\bar{T}_2 = T_{21}\,\bar{e}_1 - T_2\,\bar{e}_2 + T_{23}\,\bar{e}_3 \qquad (2.3.1)$$

$$\bar{M}_2 = M_2\,\bar{e}_1 - M_{21}\,e_2$$

and along β curve,

$$\bar{T}_1 = -T_1\,\bar{e}_1 - T\ \ \bar{e}_2 + T_{13}\,\bar{e}_3 \qquad (2.3.2)$$

$$\bar{M}_1 = -M_{12}\,\bar{e}_1 - M_1\,\bar{e}_2$$

where T_1 and T_2 = normal forces/unit length

$T_{12}, T_{21}, T_{13}, T_{23}$ = shearing forces/unit length

M_1 and M_2 = moments/unit length

M_{12} and M_{21} = twisting moments/unit length

These forces and moments are expressed in term of normal and shearing stresses (see figure 6) as given below,

$$T_1 = \int_{-h/2}^{+h/2} \sigma_1 \left(1 + \frac{z}{R_2}\right) dz$$

$$T_2 = \int_{-h/2}^{+h/2} \sigma_2 \left(1 + \frac{z}{R_1}\right) dz$$

$$T_{12} = \int_{-h/2}^{+h/2} \sigma_{12} \left(1 + \frac{z}{R_2}\right) dz$$

$$T_{21} = -\int_{-h/2}^{+h/2} \sigma_{21} \left(1 + \frac{z}{R_1}\right) dz$$

$$T_{23} = -\int_{-h/2}^{+h/2} \sigma_{32} \left(1 + \frac{z}{R_2}\right) dz$$

$$T_{13} = -\int_{-h/2}^{+h/2} \sigma_{31} \left(1 + \frac{z}{R_1}\right) dz$$

$$M_1 = -\int_{-h/2}^{+h/2} \sigma_1 \, z(1 + \frac{z}{R_2}) \, dz$$

$$M_2 = -\int_{-h/2}^{+h/2} \sigma_2 \, z(1 + \frac{z}{R_1}) \, dz$$

$$M_{12} = \int_{-h/2}^{+h/2} \sigma_{12} \, z(1 + \frac{z}{R_2}) \, dz$$

$$M_{21} = -\int_{-h/2}^{+h/2} \sigma_{21} \, z(1 + \frac{z}{R_1}) \, dz \qquad (2.3.3)$$

where σ_i' and σ_{ij} are normal and shear stresses respectively and 'h' is the thickness of the shell.

By substituting the values for the stresses we get,

$$T_1 = C_1 \left(\frac{e_1 - \cot x \, \omega + \nu e_2}{\sin x} \right)$$

$$T_2 = C_1 \left(\frac{e_2 - \cot x \, \omega + \nu e_1}{\sin x} \right)$$

$$T_{12} = -T_{21} = C_2 \left\{ \frac{1 + \cos^2 x}{\sin^2 x} \, \omega - \cot x(e_1 + e_2) - \nu(\omega + \cot x \, (e_1 + e_2)) \right\}$$

$$M_1 = C_3 \left(\frac{k_1 + \nu k_2}{\sin x} \right)$$

$$M_2 = C_3 \left(\frac{k_2 + \nu k_1}{\sin \chi} \right)$$

$$M_{12} = - C_4 \frac{\tau^2}{\sin \chi}$$

$$M_{21} = - C_4 \left(\frac{\tau' + \frac{\omega}{2R_1}}{\sin \chi} \right) \tag{2.3.4}$$

where ν = Poisson's ratio

h = thickness of the shell

$$C_1 = \frac{Eh}{(1 - \nu^2)}$$

$$C_2 = \frac{Eh}{2(1 - \nu^2)}$$

$$C_3 = - \frac{Eh^3}{12(1 - \nu^2)}$$

$$C_4 = \frac{Eh^3}{12(1 + \nu)} \tag{2.3.5}$$

E = Young's Modulus of Elasticity

upon substitution of the values of strain from equations (2.2.3 to 2.2.6), we get,

$$T_1 = C_1 \Big\{ \frac{B}{D}\frac{\partial u}{\partial x} - \frac{w}{2R_1}\frac{AB}{D}\left(1 + \frac{z_x^2 z_y^2}{D^2}\right)$$

$$- \frac{z_x z_y}{BD}(1 - \nu)\frac{\partial u}{\partial y} + \frac{z_x z_y^2 z_{xx} B}{A^2 D^3} u$$

$$+ \frac{\nu A}{D}\frac{\partial v}{\partial y} \Big\}$$

$$T_2 = C_1 \Big\{ \frac{A}{D}\frac{\partial v}{\partial y} - \frac{z_x z_y^2 z_{xx}}{A^2 D^3} B \cdot u - \frac{z_x z_y}{AD}(1 - \nu)\frac{\partial v}{\partial x}$$

$$- \frac{AB\, z_x^2 z_y^2}{R_1 D^3}\left(1 + \nu \frac{D^2}{z_x^2 z_y^2}\right) w + \frac{\nu B}{D}\frac{\partial u}{\partial x} \Big\}$$

$$T_{12} = - T_{21} = C_2 \Big\{ \frac{\partial u}{\partial y}\left[\frac{A}{D} - \nu\left(\frac{D}{AB^2} + \frac{z_x^2 z_y^2}{AD\, B^2}\right)\right]$$

$$+ \frac{\partial v}{\partial x}\left[\frac{B}{D} - \nu\left(\frac{D}{A^2 B} + \frac{z_x^2 z_y^2}{DA^2 B}\right)\right]$$

$$+ u \frac{z_y z_{xx}}{A^3 D}\left(\frac{A^2 B^2}{D^2} + \frac{z_x^2 z_y^2}{D^2} - \nu \right)$$

$$+ w \frac{2 z_x z_y}{D^3 R_1}\left(D^2 + z_x^2 z_y^2\right)$$

$$- \frac{\partial u}{\partial x} \cdot \frac{z_x z_y (1 + \nu)}{AD}$$

$$- \frac{\partial v}{\partial y} \cdot \frac{z_x z_y}{BD}(1 + \nu) \Big\}$$

$$M_1 = C_3 \Big\{ - \frac{\partial w}{\partial y} \cdot \frac{z_y z_{xx}}{A^2 D^2} \left(1 + \frac{z_x^2 \, z_y^2}{D^2}\right)$$

$$- \frac{z_x z_y}{D^2} (1 + \nu) \cdot \frac{\partial^2 w}{\partial x \partial y}$$

$$- B^2 \frac{z_x z_{xx}}{R_1 A D^4} (D^2 + 2A^2) - \frac{B^2 z_x z_{xx}}{D^4} \cdot \frac{\partial w}{\partial x}$$

$$+ \frac{B^2 A}{R_1 D^2} \frac{\partial u}{\partial x} + \frac{B^2}{D^2} \frac{\partial u}{\partial x} + \frac{B^2}{D^2} \frac{\partial^2 w}{\partial x^2}$$

$$+ \nu \frac{A^2}{D^2} \frac{\partial^2 w}{\partial y^2} - \frac{\nu z_x z_y A}{R_1 D^2} \frac{\partial u}{\partial y} \Big\}$$

$$M_2 = C_3 \Big\{ \frac{A^2}{D^2} \frac{\partial^2 w}{\partial y^2} - \frac{z_x z_y A}{R_1 D^2} \frac{\partial u}{\partial y} - \frac{z_x z_y}{D^2} (1 + \nu) \frac{\partial^2 w}{\partial x \partial y}$$

$$- \nu \frac{\partial w}{\partial y} \frac{z_y z_{xx}}{A^2 D^3} \left(1 + \frac{z_x^2 \, z_y^2}{D^2}\right)$$

$$- \nu \frac{B^2 z_x z_{xx}}{R_1 A D^4} (D^2 + 2A^2) \, u - \nu \frac{B^2}{D^4} z_x z_{xx} \frac{\partial w}{\partial x}$$

$$+ \frac{B^2 \nu A}{R_1 D^2} \frac{\partial u}{\partial x} + \nu \frac{B^2}{D^2} \frac{\partial^2 w}{\partial x^2} \Big\}$$

$$M_{12} = C_4 \Big\{ - \frac{A \, z_x z_y}{B D^2} \frac{\partial^2 w}{\partial y^2} + \frac{A^2 B}{D R_1} \frac{\partial u}{\partial y} + \frac{AB}{D^2} \frac{\partial^2 w}{\partial x \partial y} \Big\}$$

$$M_{21} = -C_4 \left\{ \frac{AB}{D^2} \frac{\partial^2 w}{\partial x \partial y} + \frac{B\, z_y^2\, z_x\, z_{xx}}{AD^4} \frac{\partial w}{\partial y} \right.$$

$$+ \frac{B\, z_y^2\, z_x\, z_{xx}}{R_1 A^2 D^4} (3A^2 + z_y^2)$$

$$- \frac{z_x z_y\, B}{R_1 D^2} \frac{\partial u}{\partial x} - \frac{z_x z_y\, B}{AD^2} \frac{\partial^2 w}{\partial x^2}$$

$$\left. + \frac{z_x^2\, z_y\, z_{xx}}{AD^4}\; B \cdot \frac{\partial w}{\partial x} + \frac{1}{R_1 B} \frac{\partial u}{\partial y} \right\}$$

2.4 Equilibrium Equations:

Let the external loading and external moments at any point be given by,

$$\bar{P} = P_1\, \bar{e}_1 + P_2\, \bar{e}_2 - P_3\, \bar{e}_3 \tag{2.4.1}$$

$$\bar{M}_e = \bar{M}_{e_1}\, \bar{e}_1 + M_{e2}\, \bar{e}_2 \tag{2.4.2}$$

The equation of equilibrium are given by |30|,

$$\frac{1}{\sin\chi} \frac{\partial}{\partial x}\left[B(T_1 + \cos\chi\; T_{12})\right] - \frac{B^2}{A}(\Gamma_{11}^{2})\sin\chi\; T_{12}$$

$$- \frac{1}{\sin\chi} \frac{\partial}{\partial y}\left[A(T_{21} - \cos\chi\; T_2)\right] - B(\Gamma_{12}^{2})\sin\chi\, T_2$$

$$- \frac{AB}{\sin\chi}\left(\frac{T_{13}}{R_1} - \frac{T_{23}}{R_{12}}\right) + AB(P_1 + \cos\chi\; P_2) = 0$$

$$\frac{1}{\sin\chi}\frac{\partial}{\partial x}\left[B(T_{12}+\cos\chi\;T_1)\right]-A\,(\Gamma_{12}^{1})\sin\chi\;T_1$$

$$+\frac{1}{\sin\chi}\frac{\partial}{\partial y}\left[A(T_2-\cos\chi\;T_{21})\right]+\frac{A^2}{B}\Gamma_{22}^{1}\sin\chi$$

$$-\frac{AB}{\sin\chi}\left(\frac{T_{23}}{R_2}-\frac{T_{13}}{R_{12}}\right)+AB(P_2+\cos\chi\;P_1)=0$$

$$AB\left(\frac{T_1}{R_1}+\frac{T_2}{R_2}+\frac{T_{21}-T_{12}}{R_{12}}\right)+\frac{\partial}{\partial x}(B\,T_{13})+\frac{\partial}{\partial y}(A\,T_{23})$$

$$AB\sin\chi\;P_3=0$$

$$+\frac{1}{\sin\chi}\frac{\partial}{\partial x}\left[B(M_{12}+\cos\chi\;M_1)\right]-\frac{B^2}{A}\Gamma_{11}^{2}\sin\chi\;M_1$$

$$-\frac{1}{\sin\chi}\frac{\partial}{\partial y}\left[A(M_2-\cos\;M_{21})\right]-B\,\Gamma_{12}^{2}\sin\chi\;M_{21}$$

$$+AB\;T_{23}+AB(M_{e1}+\cos\chi\;M_{e2})=0$$

$$\frac{1}{\sin\chi}\frac{\partial}{\partial x}\left[B(M_1+\cos\chi\;M_{12})\right]-A\,\Gamma_{12}^{1}\sin\chi\;M_{12}$$

$$+\frac{1}{\sin\chi}\frac{\partial}{\partial y}\left[A(M_{21}-\cos\chi\;M_2)\right]-\frac{A^2}{B}\Gamma_{22}^{1}\sin\chi\;M_2$$

$$-AB\;T_{13}+AB(M_{e2}+\cos\chi\;M_{e1})=0.$$

$$\sin\chi\;(T_{12}+T_{21})+\frac{M_{12}}{R_1}+\frac{M_{21}}{R_2}+\frac{M_2-M_1}{R_{12}}=0$$

(2.4.3)

The last of the above equations remains an identity. Upon substitution of christoffel's symbols, we obtain,

$$AB \frac{\partial T_1}{\partial x} + z_x z_y \frac{\partial T_{12}}{\partial x} - A^2 \frac{\partial T_{21}}{\partial y} + \frac{A}{B} z_x z_y \frac{\partial T_2}{\partial y}$$

$$- \frac{A^2 B}{R_1} T_{13} + AD\, P_1 + \frac{D\, z_x z_y}{B} P_2 = 0$$

$$A^3 B \frac{\partial T_{12}}{\partial x} + z_y z_{xx} T_1 + A^2 z_y z_x \frac{\partial T_1}{\partial x} - \frac{A^3}{B} z_x z_y \frac{\partial T_{21}}{\partial y}$$

$$A^3 D\, P_2 + \frac{A^2 D}{B} z_x z_y P_1 = 0$$

$$\frac{AB}{R_1} T_1 + B \frac{\partial T_{13}}{\partial x} + A \frac{\partial T_{23}}{\partial y} + DP_3 = 0$$

$$AB \frac{\partial M_{12}}{\partial x} + z_x z_y \frac{\partial M_1}{\partial x} - A^2 \frac{\partial M_2}{\partial y} + \frac{A}{B} z_x z_y \frac{\partial M_{21}}{\partial y}$$

$$+ AD\, T_{23} + AD\, M_{e1} + \frac{D}{B} z_x z_y M_{e2} = 0$$

$$A^3 B \frac{\partial M_1}{\partial x} + z_y z_{xx} M_{12} + A^2 z_x z_y \frac{\partial M_{12}}{\partial x} + A^4 \frac{\partial M_{21}}{\partial y}$$

$$- A^3 \frac{z_x z_y}{B} \frac{\partial M_2}{\partial y} - A^3 D\, T_{13} + DA^3 M_{e2}$$

$$+ \frac{A^2 D}{B} z_x z_y M_{e1} = 0$$

CHAPTER 3

SHELL EQUATIONS WHEN THE LONGITUDINAL PARABOLIC CURVE IS SHALLOW

3.1 Introduction:

Most of the shell roof structures come within the category of shallow shells where the height to length ratio is $\leq 1/5$. It should be noted that the theory used in a shallow shell theory, is based on assumption that the squares of the first order derivatives of surface functions, i.e., z_x^2, z_y^2, $z_x z_y$, *are negligible in comparison with unity.* This limitation on the application of the shallow shell theory has been examined by E. Reissner [12] who states as a general rule that the shallow shell theory will be more than sufficiently accurate as long as z_x, $z_y \leq 1/8$ and often accurate enough for practical purposes as long as z_x, $z_y \leq 1/2$.

In the case of shell under study, z_x^2, and z_{xx}^2 have been neglected but $z_x z_y$ is kept as such since it is not shallow in the transverse direction. The products of $1/R_1$ with u and v are neglected in comparison with w in the strain-displacement relation. Value of $\sin\chi$ is approximated to 1 whereas $\cos\chi$ is approximated to $z_x z_y/B$, the derivative of which is $z_{xx} z_y/B$ which is kept in tact as only the square

of surface function derivatives are omitted. Under those concessions, we can fairly say that the formulation and analysis of the shell under study will be accurate enough for values of $H/L \leq 1/5$.

3.2 Stress Displacement Relations and Equillibrium Equations:

Under the above approximations, the stress displacement relations (2.3.6) are reduced to,

$$T_1 = C_1 \left\{ \frac{\partial u}{\partial x} - \frac{w}{R_1} - \frac{z_x z_y}{B^2}(1 - \nu) \frac{\partial u}{\partial y} + \frac{\nu}{B} \frac{\partial v}{\partial y} \right\}$$

$$T_2 = C_1 \left\{ \frac{1}{B} \frac{\partial v}{dy} - \frac{z_x z_y}{B} (1 - \nu) \frac{\partial v}{\partial x} + \nu \frac{\partial u}{\partial x} - \frac{\nu w}{R_1} \right\}$$

$$T_{12} = -T_{21} = C_2 \left\{ \frac{1}{B} (1 - \nu) \frac{\partial u}{\partial y} + (1 - \nu) \frac{\partial v}{\partial x} \right.$$

$$+ 2 \frac{z_x z_y}{BR_1} w - \frac{z_x z_y}{B} (1 + \nu) \frac{\partial u}{\partial x}$$

$$\left. - \frac{z_x z_y}{B^2} (1 + \nu) \frac{\partial v}{\partial y} \right\}$$

$$M_{12} = C_4 \left\{ - \frac{z_x z_y}{B^3} \frac{\partial^2 w}{\partial y^2} + \frac{1}{B} \frac{\partial^2 w}{\partial x \partial y} \right\}$$

$$M_{21} = - C_4 \left\{ \frac{1}{B} \frac{\partial^2 w}{\partial x \partial y} - \frac{z_x z_y}{B} \frac{\partial^2 w}{\partial x^2} \right\}$$

$$M_1 = C_3 \left\{ - \frac{z_y z_{xx}}{B^2} \frac{\partial w}{\partial y} - \frac{z_x z_y (1+\nu)}{B^2} \frac{\partial^2 w}{\partial x \partial y} + \frac{\partial^2 w}{\partial x^2} + \frac{\nu}{B^2} \frac{\partial^2 w}{\partial y^2} \right\}$$

$$M_2 = C_3 \left\{ \frac{1}{B^2} \frac{\partial^2 w}{\partial y^2} - \frac{z_x z_y (1+\nu)}{B^2} \frac{\partial^2 w}{\partial x \partial y} + \nu \frac{\partial^2 w}{\partial x^2} - \nu \frac{z_y z_{xx}}{B^2} \frac{\partial w}{\partial y} \right\} \quad (3.2.1)$$

In the first equillibrium equation, the product of B/R_1 with T_{13} is neglected and putting $T_{12} = -T_{21}$ we get,

$$B \frac{\partial T_1}{\partial x} + z_x z_y \frac{\partial T_{12}}{\partial x} + \frac{z_x z_y}{B} \frac{\partial T_2}{\partial y} + BP_1 + z_x z_y P_2 = 0$$

$$B \frac{\partial T_{12}}{\partial x} + z_y z_{xx} T_1 + z_x z_y \frac{\partial T_1}{\partial x} + \frac{z_x z_y}{B} \frac{\partial T_{12}}{\partial y} + BP_2 + z_x z_y P_1 = 0$$

$$\frac{B}{R_1} T_1 + B \frac{\partial T_{13}}{\partial x} + \frac{\partial T_{23}}{\partial y} + BP_3 = 0$$

$$B \frac{\partial M_{12}}{\partial x} + z_x z_y \frac{\partial M_1}{\partial x} - \frac{\partial M2}{\partial y} + \frac{z_x z_y}{B} \frac{\partial M_{21}}{\partial y} + BT_{23} + BM_{e1} + z_x z_y M_{e2} = 0$$

$$B \frac{\partial M_1}{\partial x} + z_y z_{xx} M_{12} + z_x z_y \frac{\partial M_{12}}{\partial x} + \frac{\partial M_{21}}{\partial y}$$

$$- \frac{z_x z_y}{B} \frac{\partial M_2}{\partial y} - BT_{13} + BM_{e2} + z_x z_y M_{e1} = 0 \tag{3.2.2}$$

3.3 Loading on the Structure:

The loading considered for the structure are,

a) Uniformly distributed load in plan:

For uniformly distributed load, the load vector is $- P_z \bar{k}$ where P_z is the magnitude of the load. From equation (2.1.4)

$$\text{i.e. } P_1 = - \frac{P_z \cdot z_x A}{D^2} ; \quad P_2 = \frac{- P_z z_y B}{D^2} ; \; P_3 = \frac{P_z}{D} \tag{3.3.1}$$

b) Wind loads:

The analysis of structure for wind loads is done as per IS Codes |26|. The loading condition for a single fold is given in Table I. For multiple folds, the loading remains the same to the windward span and the leeward slope of the leeward span as per Table I. On all other roof slopes, account need be taken of only the effect of wind drag

(see Fig. 7) For wind loads acting normal to the longitudinal direction, the load vector can be written as, $-P_w\bar{j}$, where P_w is the intensity of wind pressure. From equation (2.1.4),

$$P_1 = P_w \frac{z_y z_x A}{D^2} \; ; \quad P_2 = -\frac{P_w A^2 B}{D^2} \; ; \quad P_3 = -\frac{P_w z_y}{D} \tag{3.3.2}$$

3.4 <u>Reduction of Equillibrium Equations:</u>

The five equillibrium equations (3.2.2) are reduced to three equations in u, v, w directions respectively by substituting for T_{13} and T_{23}. These equations in forces and moments when expressed in displacements u, v, w, we get,

$$\begin{aligned}
&-z_x z_y^2 z_{xx} \frac{(1+\nu)}{B} \frac{\partial u}{\partial x} - 2 z_y z_{xx} \frac{(1-\nu)}{B} \frac{\partial u}{\partial y} + B\left[2 - z_x^2 z_y^2 \frac{(1+\nu)}{B^2}\right] \frac{\partial^2 u}{\partial x^2} \\
&+ \frac{(1-\nu)}{B} \frac{\partial^2 u}{\partial y^2} - 2 z_x z_y \frac{(1-\nu)}{B} \frac{\partial^2 u}{\partial x \partial y} + z_x z_y (1-\nu) \frac{\partial^2 v}{\partial x^2} - z_x z_y^2 z_{xx} \frac{(1+\nu)}{B^2} \frac{\partial v}{\partial y} \\
&+ z_x z_y \frac{(1-\nu)}{B^2} \frac{\partial^2 v}{\partial y^2} - \left[z_x^2 z_y^2 \frac{(3-\nu)}{B^2} - (1+\nu)\right] \frac{\partial^2 v}{\partial x \partial y} - \frac{2B}{R_1} \frac{\partial w}{\partial x} \\
&+ 2 z_x z_y \frac{(1-\nu)}{BR_1} \frac{\partial w}{\partial y} + \frac{B}{C_2} P_1 + \frac{z_x z_y}{C_2} P_2 = 0
\end{aligned} \tag{3.4.1}$$

$$Z_y Z_{xx} (1-\nu) \frac{\partial u}{\partial x} - 2 Z_y^2 Z_x Z_{xx} \frac{(1-\nu)}{B^2} \frac{\partial u}{\partial y} + Z_x Z_y (1-\nu) \frac{\partial^2 u}{\partial x^2}$$

$$+ Z_x Z_y \frac{(1-\nu)}{B^2} \frac{\partial^2 u}{\partial y^2} - \left[Z_x^2 Z_y^2 \frac{(3-\nu)}{B^2} - (1+\nu) \right] \frac{\partial^2 u}{\partial x \partial y}$$

$$+ B(1-\nu) \frac{\partial^2 v}{\partial x^2} - Z_y Z_{xx} \frac{(1-\nu)}{B} \frac{\partial v}{\partial y} + \left[\frac{2}{B} - Z_x^2 Z_y^2 \frac{(1+\nu)}{B^3} \right] \frac{\partial^2 v}{\partial y^2}$$

$$- 2 Z_x Z_y \frac{(1-\nu)}{B} \frac{\partial^2 v}{\partial x \partial y} + \frac{2}{R_1} \left[\frac{Z_x^2 Z_y^2}{B^2} - \nu \right] \frac{\partial w}{\partial y}$$

$$+ \frac{B}{C_2} P_2 + \frac{Z_x Z_y}{C_2} P_1 = 0$$

$$(3.4.2)$$

$$- \frac{12}{h^2} \left(\frac{B}{R_1} \frac{\partial u}{\partial x} - \frac{B w}{R_1^2} - Z_x Z_y \frac{(1-\nu)}{B R_1} \frac{\partial u}{\partial y} + \frac{\nu}{R_1} \frac{\partial v}{\partial y} \right)$$

$$- 6 \frac{Z_y Z_{xx}}{B} \frac{\partial^3 w}{\partial x^2 \partial y} + 8 \frac{Z_y^2 Z_x Z_{xx}}{B^3} \frac{\partial^3 w}{\partial x \partial y^2} + B \frac{\partial^4 w}{\partial x^4}$$

$$- 4 \frac{Z_x Z_y}{B} \frac{\partial^4 w}{\partial x^3 \partial y} - 4 \frac{Z_x Z_y}{B^3} \frac{\partial^4 w}{\partial x \partial y^3} + Z_y^2 Z_{xx}^2 \frac{(2-\nu)}{B^3} \frac{\partial^2 w}{\partial y^2}$$

$$- 2 \frac{Z_y Z_{xx}}{B^3} \frac{\partial^3 w}{\partial y^3} + \frac{1}{B^3} \frac{\partial^4 w}{\partial y^4} + 2 \left[\frac{1}{B} + 2 \frac{Z_x^2 Z_y^2}{B^3} \right] \frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{B}{C_3} P_3 = 0.$$

$$(3.4.3)$$

The coordinates and displacements are non-dimensionalised as given below,

$$u = \bar{u}\, h \; ; \quad v = \bar{v}\, h \; ; \quad w = \bar{w}\, h \; ;$$

$$x = X\, L \; ; \quad y = Y\, b \; ;$$

$$\frac{H}{L} = a_1 \; ; \quad \frac{b}{L} = a_2 \; ; \quad \frac{h}{L} = a_3 \; ;$$

$$P = \frac{P_z L^4}{\bar{D}\, h} \quad \text{where} \quad \bar{D} = Eh^3/12(1 - \nu^2)$$

$$(3.4.4)$$

Using relation (3.3;2), the equillibrium equations are written as,

$$\begin{aligned} E_1(\bar{u},\bar{v},\bar{w},x,y) \equiv\ & A_1 x \frac{\partial \bar{u}}{\partial x} + A_2 \frac{\partial \bar{u}}{\partial y} + A_3 \frac{\partial^2 \bar{u}}{\partial x^2} + A_{14} x^2 \frac{\partial^2 \bar{u}}{\partial x^2} + A_5 \frac{\partial^2 \bar{u}}{\partial y^2} \\ & + A_6 x \frac{\partial^2 \bar{u}}{\partial x \partial y} + A_7 x \frac{\partial^2 \bar{v}}{\partial x^2} + A_8 x \frac{\partial \bar{v}}{\partial y} + A_9 x \frac{\partial^2 \bar{v}}{\partial y^2} \\ & + A_{10} x^2 \frac{\partial^2 \bar{v}}{\partial x \partial y} + A_{11} \frac{\partial^2 \bar{v}}{\partial x \partial y} + A_{12} \frac{\partial \bar{w}}{\partial x} + A_{13} x \frac{\partial \bar{w}}{\partial y} \\ & + A_{14} x P = 0. \end{aligned} \qquad (3.4.5)$$

$$\begin{aligned} E_2(\bar{u},\bar{v},\bar{w},x,y) \equiv\ & B_1 \frac{\partial \bar{u}}{\partial x} + B_2 x \frac{\partial \bar{u}}{\partial y} + B_3 x \frac{\partial^2 \bar{u}}{\partial x^2} + B_4 x \frac{\partial^2 \bar{u}}{\partial y^2} \\ & + B_5 x^2 \frac{\partial^2 \bar{u}}{\partial x \partial y} + B_6 \frac{\partial^2 \bar{u}}{\partial x \partial y} + B_7 \frac{\partial^2 \bar{v}}{\partial x^2} + B_8 \frac{\partial \bar{u}}{\partial y} \\ & + B_9 \frac{\partial^2 \bar{v}}{\partial x^2} + B_{10} x^2 \frac{\partial^2 \bar{v}}{\partial x^2} + B_{11} x \frac{\partial^2 \bar{v}}{\partial x \partial y} + B_{12} x^2 \frac{\partial \bar{w}}{\partial y} \\ & + B_{15} \frac{\partial \bar{w}}{\partial y} + B_{13} P + B_{14} x^2 P = 0. \end{aligned} \qquad (3.4.6)$$

$$E_3(\bar{u},\bar{v},\bar{w},x,y) \equiv C_1 \frac{\partial \bar{u}}{\partial x} + C_2 \bar{w} + C_3 \, x \frac{\partial \bar{u}}{\partial y} + C_4 \frac{\partial \bar{v}}{\partial y} + C_5 \frac{\partial^3 \bar{w}}{\partial x^2 \partial y}$$
$$+ C_6 \, x \frac{\partial^3 \bar{w}}{\partial x \partial y^2} + C_7 \frac{\partial^4 \bar{w}}{\partial x^4} + C_8 \, x \frac{\partial^4 \bar{w}}{\partial x^3 \partial y} + C_9 \, x \frac{\partial^4 \bar{w}}{\partial x \partial y^3}$$
$$+ C_{10} \frac{\partial^2 \bar{w}}{\partial y^2} + C_{11} \frac{\partial^3 \bar{w}}{\partial y^3} + C_{12} \frac{\partial^4 \bar{w}}{\partial y^4} + C_{13} \frac{\partial^4 \bar{w}}{\partial x^2 \partial y^2}$$
$$+ C_{14} \, x^2 \frac{\partial^4 \bar{w}}{\partial x^2 \partial y^2} + C_{15} \, P = 0 \qquad (3.4.7)$$

The constants A's, B's, C's are given in the Appendix A.

The stress-displacement relations in non-dimensionalised form become,

$$\frac{T_1}{C_1} = G_{11} \frac{\partial \bar{u}}{\partial X} + G_{12} \bar{w} + G_{13} \, X \frac{\partial \bar{u}}{\partial Y} + G_{14} \frac{\partial \bar{v}}{\partial Y}$$

$$\frac{T_2}{C_1} = G_{21} \frac{\partial \bar{v}}{\partial Y} + G_{22} \, X \frac{\partial \bar{v}}{\partial X} + G_{23} \frac{\partial \bar{u}}{\partial X} + G_{24} \bar{w}$$

$$\frac{T_{12}}{C_1} = G_{31} \frac{\partial \bar{u}}{\partial Y} + G_{32} \frac{\partial \bar{v}}{\partial X} + G_{33} \, X \bar{w} + G_{34} \, X \frac{\partial \bar{u}}{\partial X} + G_{35} \, X \frac{\partial \bar{v}}{\partial Y}$$

$$\frac{M_1 b}{D} = G_{41} \frac{\partial \bar{w}}{\partial Y} + G_{42} \, X \frac{\partial^2 \bar{w}}{\partial x \partial Y} + G_{43} \frac{\partial^2 \bar{w}}{\partial X^2} + G_{44} \frac{\partial^2 \bar{w}}{\partial Y^2}$$

$$\frac{M_2 b}{\bar{D}} = G_{51} \frac{\partial^2 \bar{w}}{\partial Y^2} + G_{52}\, X \frac{\partial^2 \bar{w}}{\partial X \partial Y} + G_{53} \frac{\partial^2 \bar{w}}{\partial X^2} + G_{54} \frac{\partial \bar{w}}{\partial Y}$$

$$\frac{M_{12} b}{\bar{D}} = G_{61} \frac{\partial^2 w}{\partial Y^2} + G_{62} \frac{\partial^2 \bar{w}}{\partial X \partial Y}$$

$$\frac{M_{21} b}{\bar{D}} = G_{71} \frac{\partial^2 \bar{w}}{\partial X \partial Y} + G_{72} \frac{\partial^2 \bar{w}}{\partial X^2} \qquad (3.4.8)$$

The coefficients G_{ij} are given in Appendix A.

CHAPTER 4

BOUNDARY CONDITIONS

4.1 Transverse Edge Boundary Conditions and Displacement Functions:

The '*Kontorovitch method*' is applied for the solution of the problem. The functions governing the behaviour, in the longitudinal direction, of displacements u, v, w are assumed. By the method of virtual work, the equations are reduced to ordinary differential equations of unknown functions in the transverse direction. Assuming,

$$\bar{u} = \sum_{m=1}^{M} u_m^*(Y)\ f_m(X)$$

$$\bar{v} = \sum_{m=1}^{M} v_m^*(Y)\ \phi_m(X)$$

$$\bar{w} = \sum_{m=1}^{M} w_m^*(Y)\ \psi_m(X)$$

The equilibrium equations are integrated in the X-direction by virtual work principle.

$$\int_{-1/2}^{1/2} E_1(\bar{u}, \bar{v}, \bar{w}, X, Y)\ \delta(\bar{u})\ dX = 0$$

$$\int_{-1/2}^{1/2} E_2(\bar{u},\ \bar{v},\ \bar{w},\ X,\ Y)\ \delta(\bar{v})\ dX \ = \ 0$$

$$\int_{-1/2}^{1/2} E_3(\bar{u},\ \bar{v},\ \bar{w},\ X,\ Y)\ \delta(\bar{w})\ dX \ = \ 0 \qquad (4.1.1)$$

Expanding the displacements in terms of displacement functions,

$$\int_{-1/2}^{1/2} E_1(\bar{u},\ \bar{v},\ \bar{w},\ X,\ Y)\ \delta\ u_m^*(Y)\ \ f_m(X)\ dX \ = \ 0$$

$$\int_{-1/2}^{1/2} E_2(\bar{u},\ \bar{v},\ \bar{w},\ X,\ Y)\ \delta\ v_m^*(Y)\ \ \phi_m(X)\ dX \ = \ 0$$

$$\int_{-1/2}^{1/2} E_3(\bar{u},\ \bar{v},\ \bar{w},\ X,\ Y)\ \delta\ w_m^*(Y)\ \ \psi_m(X)\ dX \ = \ 0$$

$$m = 1, \ldots M \qquad (4.1.2)$$

Thus giving,

$$\delta u_k^* \cdot \bar{E}_1(u_k^*,\ v_k^*,\ w_k^*,\ Y) \ = \ 0$$

$$\delta v_k^* \cdot \bar{E}_2(u_k^*,\ v_k^*,\ w_k^*,\ Y) \ = \ 0$$

$$\delta w_k^* \cdot \bar{E}_3(u_k^*,\ v_k^*,\ w_k^*,\ Y) \ = \ 0$$

$$k = 1, \ldots . M \qquad (4.1.3)$$

4.2 Stress Displacement Relations:

The stress displacement relations are expressed in matrix form.

$$
\begin{bmatrix} \frac{T_1}{C_1} \\ \frac{T_2}{C_1} \\ \frac{T_{12}}{C_1} \\ \frac{M_1 b}{D} \\ \frac{M_2 b}{D} \\ \frac{M_{12} b}{D} \\ \frac{M_{21} b}{D} \end{bmatrix}
=
\begin{bmatrix}
G_{11} f_i'(x) & 0 & G_{12} \psi_i(x) & 0 & G_{13} \times f_i(x) & G_{14} \phi_i(x) & 0 \\
G_{23} f_i'(x) & G_{22} \times \phi_i'(x) & G_{24} \psi_i(x) & 0 & 0 & G_{21} \phi_i(x) & 0 \\
G_{34} \times f_i'(x) & G_{32} \phi_i'(x) & G_{33} \times \psi_i(x) & 0 & G_{31} f_i(x) & G_{35} \times \phi_i(x) & 0 \\
0 & 0 & G_{43} \psi_i''(x) & G_{42} \times \psi_i'(x) + G_{41} \psi_i(x) & 0 & 0 & G_{44} \psi_i(x) \\
0 & 0 & G_{53} \psi_i''(x) & G_{54} \psi_i(x) + G_{52} \times \psi_i'(x) & 0 & 0 & G_{51} \psi_i'(x) \\
0 & 0 & 0 & G_{62} \psi_i'(x) & 0 & 0 & G_{61} \times \psi_i(x) \\
0 & 0 & G_{72} \times \psi_i''(x) & G_{71} \psi_i'(x) & 0 & 0 & 0
\end{bmatrix}
\begin{bmatrix} u_i^* \\ v_i^* \\ w_i^* \\ \frac{dw^*}{dy_i} \\ \frac{du^*}{dy_i} \\ \frac{du^*}{dy_i} \\ \frac{d^2 w^*}{dy_i^2} \end{bmatrix}
$$

$$F = \bar{T} \, \bar{u} \qquad (4.2.1)$$

4.3 Fixed Boundary Condition:

$$\bar{u} = 0$$

$$\bar{v} = 0$$

$$\bar{w} = 0$$

$$\frac{\partial \bar{w}}{\partial X} = 0$$

The functions assumed are,

$$f_m(X) = \sin(2m\pi X) \qquad m = 1,2, \ldots M$$

$$\phi_m(X) = \cos(m\pi X) \qquad m = 1,3, \ldots M$$

$$\psi_m(X) = (\cos(2m\pi X) - (-1)^m) \quad m = 1,2,\ldots M \tag{4.3.1}$$

4.4 Hinged Boundary Condition at Supports:

At the hinged boundary,

$$\bar{u} = 0$$

$$\bar{v} = 0$$

$$\bar{w} = 0$$

$$M_1 \rightarrow \frac{\partial^2 \bar{w}}{\partial X^2} = 0$$

And the displacement functions can be written as,

$$f_m(X) = \sin 2m\pi X \qquad m = 1,2, \ldots M$$

$$\phi_m(X) = \cos m\pi X \qquad m = 1,3, \ldots N$$

$$\psi_m(X) = \cos m\pi X \qquad m = 1,3, \ldots N$$

(4.4.1)

Taking M terms in each of the series.

4.5 Equilibrium Equations and Stress Displacement Relation:

In the following only one term of the series is taken. The accuracy of this has been established in the analysis of plates and shallow shells. Taking one term, the equilibrium equation become,

$$\bar{A}_1 u^* + \bar{A}_2 \frac{du^*}{dY} + \bar{A}_3 \frac{d^2u^*}{dY^2} + \bar{A}_4 v^* + \bar{A}_5 \frac{dv^*}{dY} + \bar{A}_6 \frac{d^2v^*}{dY^2}$$

$$+ \bar{A}_7 w^* + \bar{A}_8 \frac{dw^*}{dY} + \bar{A}_9 P = 0$$

$$\bar{B}_1 u^* + \bar{B}_2 \frac{du^*}{dY} + \bar{B}_3 \frac{d^2u^*}{dY^2} + \bar{B}_4 v^* + \bar{B}_5 \frac{dv^*}{dY} + \bar{B}_6 \frac{d^2v^*}{dY^2}$$

$$+ \bar{B}_7 \frac{dw^*}{dY} + \bar{B}_8 P = 0$$

$$\bar{C}_1 u^* + \bar{C}_2 w^* + \bar{C}_3 \frac{du^*}{dY} + \bar{C}_4 \frac{dv^*}{dY} + \bar{C}_5 \frac{dw^*}{dY} + \bar{C}_6 \frac{d^3w^*}{dY^3}$$

$$+ \bar{C}_7 \frac{d^3w^*}{dY^3} + \bar{C}_8 \frac{d^4w^*}{dY^4} + C_9 P = 0 \qquad (4.5.1)$$

The coefficients $\bar{A}$'s, $\bar{B}$'s, $\bar{C}$'s are given in the Appendix A. The stress displacement relation become,

$$F_f = \bar{\bar{T}}_f \; \bar{\bar{u}} \qquad (4.5.2)$$

$$F_s = \bar{\bar{T}}_s \; \bar{u} \qquad (4.5.3)$$

Equations (4.5.2) and (4.5.3) are for fixed and hinged conditions respectively. These are obtained by substituting the proper functions $f(X)$, $\phi(X)$, $\psi(X)$ in the equation (4.2.1).

4.6 <u>Longitudinal Edge Boundary Condition</u>:

Three main categories of longitudinal edge boundary conditions are considered.

1. End Condition (E.C.): The end condition comprise hinged, fixed and free conditions of the longitudinal edge of the end units.

2. Symmetric Condition (S.C.): The symmetric conditions are formulated so that the integration path can be reduced to half provided the structure is symmetric with respect to geometry, load and end conditions.

3. Joint Conditions (J.C.): The joint conditions represent the transfer of the variables from an i^{th} plate

to j^{th} plate satisfying the compatability of deformations and forces at the joint. This holds good for any ridge or valley of the structure.

4.7 End Conditions:

1) When the longitudinal edges are fixed,

$$\bar{u} = 0$$

$$\bar{v} = 0$$

$$\bar{w} = 0$$

$$\frac{\partial \bar{w}}{\partial y} = 0 \qquad (4.7.1)$$

2) The hinged conditions are,

$$\bar{u} = 0$$

$$\bar{v} = 0$$

$$\bar{w} = 0$$

$$\frac{\partial^2 \bar{w}}{dY^2} = 0 \qquad (4.7.2)$$

3) The free boundary conditions require the forces at the boundary to vanish (see figure 8).

$$V_n = T_{23} + \frac{\partial M_{21}}{\partial x} = 0$$

$$T_2 = 0$$

$$M_2 = 0$$

$$\bar{T}_1 = \text{inplane shear} = T_{21} + M_{21} = 0$$

The force boundary condition when expressed in terms of displacements are functions of x and y. This has to be reduced. to a set of ordinary differential equations suitable for numerical integration.

There are two procedures for doing so. One method adopted is to satisfy the end force boundary conditions wherever the particular force is maximum. This does not ensure the accuracy of solution at other points. But the errors introduced by doing so can be kept minimum by taking as many number of terms in the series expansion of displacements and satisfying the force boundary conditions at that many points.

When only a single or few terms are taken the better procedure is to put virtual work boundary conditions. Accordintly,

$$\int_{-1/2}^{1/2} V_n \, \delta(\bar{w}) \, dX = 0$$

$$\int_{-1/2}^{1/2} T_2 \ \delta(\bar{v}) \ dX = 0$$

$$\int_{-1/2}^{1/2} M_2 \ \delta\left(\frac{\partial \bar{w}}{\partial Y} \right) dX = 0$$

$$\int_{-1/2}^{1/2} \bar{T}_1 \ \delta(\bar{u}) \ dX = 0$$

Writing these four equations in matrix form,

$$E \ D_1 + \bar{E} \ D_2 = 0 \qquad (4.7.3)$$

where

$$\{D_1\} = \begin{bmatrix} u^*_m \\ \cdot\cdot \\ v^*_m \\ \cdot\cdot \\ w^*_m \\ \cdot\cdot \\ \frac{dw^*}{dY} \end{bmatrix} \{D_2\} = \begin{bmatrix} \frac{du^*}{dY_m} \\ \cdot\cdot \\ \frac{dv^*}{dY_m} \\ \cdot\cdot \\ \frac{d^2w^*}{dY^2_m} \\ \cdot\cdot \\ \frac{d^2w^*}{dY^3_m} \end{bmatrix} \qquad m = 1, \ldots M \qquad (4.7.3)$$

The elements of the matrix E and $\bar{E}$ are given in Appendix B.

4.8 <u>Symmetric Conditions (S.C.)</u>:

The symmetric conditions are (see figure 8),

$$\bar{v}_i \cos\theta_i - \bar{w}_i \sin\theta_i = 0$$

$$\frac{\partial \bar{w}}{\partial Y} = 0$$

$$\bar{T}_1 = 0$$

$$V_z = \text{vertical shear} = T_2 \sin\theta_i + V_n \cos\theta_i = 0$$

If the variation of $\bar{v}$ and $\bar{w}$ are not same in the longitudinal direction, the first equation is satisfied at the point where $\bar{v}$ and $\bar{w}$ are maximum. The last equation is expressed as,

$$\int_{-1/2}^{1/2} V_z \, \delta(w_z) \, dX = 0$$

where w_z = vertical displacement = $\bar{w}_i \sec\theta_i$

These four conditions can be written as,

$$S \quad \bar{D}_1 \quad + \quad \bar{S} \quad D_2 \quad = \quad 0$$

$$\left(\frac{dw^*_m}{dY} \right) = 0 \qquad (4.8.1)$$

where

$$\{\bar{D}_1\} = \begin{Bmatrix} u^*_m \\ \cdot\cdot \\ v^*_m \\ \cdot\cdot \\ w^*_m \end{Bmatrix} \qquad m = 1, \ldots M$$

The elements of the matrices S and $\bar{S}$ are given in Appendix B.

4.9 Joint Conditions (J.C.):

The joint conditions at a ridge or valley can be expressed as (see figure 8),

$$
\begin{aligned}
u_i &= u_j \\
d_{\eta_i} &= d_{\eta_j} \\
d_{\xi_i} &= d_{\xi_j} \\
d_{\zeta_i} &= +d_{\zeta_j} \\
M_{2i} &= -M_{2j} \\
F_{\eta_i} &= +F_{\eta_j} \\
F_{\xi_i} &= F_{\xi_j} \\
\bar{T}_{1i} &= \bar{T}_{1j}
\end{aligned}
$$

The corresponding conditions written in terms of displacements and forces are,

$$
\begin{aligned}
\bar{u}_i &= \bar{u}_j \\
\bar{w}_i \sin\theta_i - \bar{v}_i \cos\theta_i &= \bar{u}_j \cos\theta_j - \bar{w}_j \sin\theta_j \\
\bar{v}_i \sin\theta_i + \bar{w}_i \cos\theta_i &= \bar{w}_j \cos\theta_j + \bar{v}_j \sin\theta_j \\
\left(\frac{d\bar{w}}{dY} \right)_i &= -\left(\frac{d\bar{w}}{dY} \right)_j
\end{aligned}
\qquad (4.9.1)
$$

$$M_{2i} = -M_{2j}$$

$$T_{2i}\cos\theta_i - V_{ni}\sin\theta_i = -T_{2j}\cos\theta_j + V_{nj}\sin\theta_j$$

$$V_{ni}\cos\theta_i + T_{2i}\sin\theta_i = V_{ni}\cos\theta_j + T_{2j}\sin\theta_j$$

$$\bar{T}_{1i} = \bar{T}_{1j} \qquad (4.9.2)$$

The equation (4.9.2) are modified as,

$$\int_{-1/2}^{1/2} M_{2i}\ \delta\left(\frac{\partial \bar{w}}{\partial Y}\right)_i dX = -\int_{-1/2}^{1/2} M_{2j}\ \delta\left(\frac{\partial \bar{w}}{\partial Y}\right)_i dX$$

$$\bar{T}_{1i} = \bar{T}_{1j} \qquad (4.9.3)$$

$$\int_{-1/2}^{1/2} (T_{2i}\cos\theta_i - V_{ni}\sin\theta_i)\ \delta(\bar{w}_i\sin\theta_i - \bar{v}_i\cos\theta_i)dX$$

$$= \int_{-1/2}^{1/2}(-T_{2j}\cos\theta_j + V_{nj}\sin\theta_j)\delta(\bar{w}_i\sin\theta_i - \bar{v}_i\cos\theta_i)dX$$

$$\int_{-1/2}^{1/2}(V_{ni}\cos\theta_i + T_{2i}\sin\theta_i)\ \delta(\bar{v}_i\sin\theta_i + \bar{w}_i\cos\theta_i)dX$$

$$= \int_{-1/2}^{1/2}(V_{nj}\cos\theta_j + T_{2j}\sin\theta_j)\ \delta(\bar{v}_j\sin\theta_j + \bar{w}_j\cos\theta_j)dX$$

$$(4.9.4)$$

Equation (4.9.4) can be written as,

$$\delta w^*_{im}(\Theta^1_{Hi}) + \delta v^*_{im}(\Theta^2_{Hi}) = \delta w^*_{im}(\Theta^1_{Hj}) + \delta v^*_{im}(\Theta^2_{Hj})$$

$$\delta w^*_{im}(\Theta^1_{Vi}) + \delta v^*_{im}(\Theta^2_{Vi}) = \delta w^*_{im}(\Theta^1_{Vj}) + \delta v^*_{im}(\Theta^2_{Vj})$$

i.e. $$\Theta^1_{Hi} + \Theta^1_{Vi} = \Theta^1_{Hj} + \Theta^1_{Vj}$$

$$\Theta^2_{Hi} + \Theta^2_{Vi} = \Theta^2_{Hj} + \Theta^2_{Vj} \qquad (4.9.5)$$

Now the equations (4.9.1)(4.9.3)(4.9.5) comprise the joint condition. Expressing in matrix form,

$$[X_i]\ [D_1]_i = [X_j]\ [D_1]_i \qquad (4.9.6)$$

$$[A_{1i}]\ [D_1]_i + [A_{2i}]\ [D_2]_i =$$

$$[A_{1j}]\ [D_1]_j + [A_{2j}]\ [D_2]_j$$

The elements of the matrix X_i, X_j, A_{1i}, A_{2i}, A_{1j}, A_{2j} are given in the Appendix B.

CHAPTER 5

NUMERICAL ANALYSIS

The numerical integration is carried out essentially using the Runge-Kutta Gill procedure. The segmentation technique is used, where in, the path is devided into a suitable number of segments so that the effect of a unit change of an unknown variable is felt at the far end of the path, thus counteracting the error propagation.

The functions to be evaluated are devided into two groups of variables namely X and T, the corresponding vectors being $\{D_1\}$ and $\{D_2\}$.

Starting Values for Segment 1:

Let the starting values be represented by $\overline{X}_{ij}$ and $\bar{T}_{ij}$ ($i = 1, \ldots n/2$; $j = 0, 1, \ldots n/2$). In the non-homogeneous solution ($j = 0$), for fixed hinged and free conditions,

$$\overline{X}_{io} = 0$$
$$\bar{T}_{io} = 0 \tag{5.1}$$

For the influence coefficient or homogeneous solution ($j = 1, \ldots n/2$), each unknown function is given a unit

increment. In the case of a free edge where the boundary condition is a matrix relation given by equation (4.7.3), the unknown vector being $\{D_2\}$, the starting values are obtained as,

$$\bar{T}_{ij} = I$$

$$\bar{X}_{ij} = -\bar{E}_{ik}^{-1} E_{kl} \bar{T}_{lj} \qquad (5.2)$$

where I is the unity matrix.

At the end of the segment 1, the functions are divided in two groups namely X_{ij} and T_{ij} ($i = 1, \ldots n/2$, $j = 0, 1, \ldots n/2$). Influence coefficients A_j are calculated so that the X group of variables at the end of this segment is zero, i.e.

$$X_{ij} A_j + X_{io} = 0$$

$$A_j = -X_{ij}^{-1} X_{jo} \qquad (5.3)$$

The modified values in T group will be,

$$t_{io} = T_{io} + T_{ij} A_j \qquad (5.4)$$

For the homogeneous solutions, the starting values B_{ij} is calculated so that the X group variables are unity at

the end of the segment, Thus,

$$X_{ij}\, B_{jk} = I$$

$$B_{ij} = X_{ij}^{-1} \qquad (5.5)$$

The modified T group variables will be,

$$t_{ij} = T_{ik}\, B_{kj} \qquad (5.6)$$

Starting values for Segment 2:

For the non-homogeneous solution, we have,

$$\bar{X}_{io} = 0$$

$$\bar{T}_{io} = t_{io}^{(1)}$$

and for homogeneous solution,

$$\bar{X}_{ij} = I$$

$$\bar{T}_{ij} = t_{ij}^{(1)} \qquad (5.7)$$

The starting values for the next segment is calculated on the same steps done for the starting values of second segment and this procedure is repeated until the last segment, except at points where a transformation of variables is necessary.

Transformation of Variables:

When a sudden jump in the integration path is met with as at the fold lines, the variables are transformed from the m^{th} to $(m + 1)^{th}$ segment. The transformation relations (4.9.6) can be written as,

$$\{D_1\}_{m+1} = [P] \{D_1\}_m \tag{5.8}$$

$$[Q][D_1]_m + [R]\{D_2\}_m = \{D_2\}_{m+1}$$

where, $[P] = [X_i]^{-1}[X_j]$

$$[Q] = [A_{2j}]^{-1}\left[[A_{1i}] - [A_{1j}][P^{-1}]\right]$$

$$[R] = [A_{2j}]^{-1}[A_{2i}]$$

The starting values for $(m + 1)^{th}$ segment are,

$$\bar{X}_{io} = 0$$

$$\bar{T}_{io} = \bar{t}_{io}^{(m)}$$

$$\bar{X}_{ij} = I$$

$$\bar{T}_{ij} = \bar{t}_{ij}^{(m)}$$

where $\bar{t}_{io}^{(m)} = R_{ik}\, t_{ko}^{(m)}$

$$\bar{t}_{ij}^{(m)} = Q_{ik}\, P_{kj} + R_{ik}\, t_{kl}^{(m)}\, P_{lj} \tag{5.9}$$

Final Boundary Condition:

For the hinged or fixed edge boundaries,

$$X_{mo} + X_{mj}\, C_j^{(l)} = 0 \qquad (5.10)$$

where m denotes the corresponding variable set to zero at the boundary and $C_j^{(l)}$ denotes the influence coefficients (correct starting values values of X group) for last segment l.

For the symmetric condition governed by equation (4.8.1), we get the corresponding boundary condition as,

$$S\left[X_{io} + X_{ij}\, C_j^{(l)}\right] + \bar{S}\left[T_{io} + T_{ij}\, C_j^{(l)}\right] = 0$$
$$i=1,\ldots m \qquad\qquad i=1,\ldots n/2$$
$$j=1,\ldots n/2$$
$$(5.11)$$

$$X_{ko} + X_{kj}\, C_j^{(l)} = 0 \quad k = m, \ldots n/2, \; j=1,\ldots n/2$$

And for the free edge boundaries governed by relation (4.7.3),

$$E\left[X_{io} + X_{ij}\, C_j^{(l)}\right] + \bar{E}\left[T_{io} + T_{ij}\, C_j^{(l)}\right] = 0$$
$$i=1,\ldots n/2$$
$$j=1,\ldots n/2$$
$$(5.12)$$

From the relations (5.10 to 5.12), the correct starting values $C_j^{(1)}$ for the last segment can be calculated in each case.

Backward Pass:

In this step, the correct starting values for X and T group of variables are evaluated as,

$$C_j^{(k-1)} = B_{jk}^{(k-1)} C_j^{(k)} + A_j^{(k-1)} \qquad (5.13)$$

$$k = 1, \ldots .2$$

The relation (5.13) holds good for all segments except where transformation of variables are needed, when,

$$C_j^{(m)} = B_{jk}^{(m)} P_{kl} C_l^{(m+1)} \qquad (5.14)$$

$C_j^{(1)}$ is the correct starting values of unknown in the first segment. In hinged and fixed edges, the other variables are known, where as in free edge, the rest of the functions are evaluated by relation (5.2).

The starting values for the T group of variables will be,

$$\bar{C}_j^{(k+1)} = t_{ij}^{(k)} C_j^{(k+1)} + T_{io}^{(k)} \qquad (5.15)$$

$$k = 1, \ldots l$$

Having known the starting values of each segment entirely, the final integration is done segment wise.

CHAPTER 6

NUMERICAL RESULTS AND CONCLUSIONS

6.1 Introduction

With this formulation for arched folded plates, where the longitudinal axis is a parabola, some other problems can also be considered viz. (a) parabolic cylinder where $\theta = 0$ (b) conventional folded plate where $H/L = 0$ (c) plates with small initial curvature.

This method of analysis is also applicable to the above variations under uniformly distributed loads though the accuracy of which needs investigation.

6.2 Results and Conclusions

The compilation and execution of the programme for the analysis of a single fold with free longitudinal edges takes approximately $6\frac{1}{2}$ minutes in IBM 7044-1401 system. Figures 9 to 23 show the behaviour of a single folded shell with fixed transverse supports and figures 24 to 36 represent shells with hinged supports.

Effect of H/L ratio

The value of normal stress T_1 depend greatly on H/L ratio. The nature of T_1 is compressive at all points

in the structure for high H/L ratio. At the central section along the ridge, the compressive stress T_1 increase as H/L ratio is reduced where as at the valley the nature of T_1 changes from compressive to tensile. At the transverse boundary, it is seen that compressive stresses develop at the valley and tensile stresses at the ridge portion. The normal stress T_1 is found to be higher at the ridge line in hinged shells and lower at the valley compared to the fixed shells. The bending stresses M_1 and M_2 increase with the decrease in H/L ratio and this effect is more predominant in hinged shells.

Effect of b/L ratio

A small increase in T_1 is noted along the ridge with decrease in b/L. There is a small rise in T_2 along the ridge. M_1 is slightly reduced at the ridge portion but remains almost unaffected at the free edge. A reduction of M_2 is observed with decrease in b/L ratio.

Effect of h/L ratio

Both in hinged and fixed shells, the bending moments M_1 and M_2 decrease with the decrease in h/L ratio. A rise in value of T_1 is observed especially at the ridge portion. The value of T_2 shows a large increase in hinged shells compared to the fixed case.

Effect of θ

The effect of variation of θ between 36^{o} and 42^{o} (which is common for the V type folded plates) is not very predominant in normal stresses T_1 and T_2. A small increase in M_1 and M_2 is noted with increase in θ in both cases.

Effect of Poisson's ratio (ν)

The variation of ν from 0 to 0.15 is found to have very little effect on normal stress T_1 but the value of T_2 is almost negligible for $\nu = 0$ and increases with ν. The value of M_1 remains same at the free edge and varies linearly at rhe central section with decrease in ν. A small increase is observed in M_2 with reduction in ν.

Discussion of the method

The function assumed for displacement w in the fixed case holds good for H/L = 0 to H/L = 1/5. For the hinged case, the cosine variation of w may not suit very well the geometry of shells with H/L = 1/5 to H/L = 1/8. For shells of low θ and H/L ratio and high b/L ratio, this method will be most apropriate and is well suited for shallow parabolic shells and plates with small initial curvature.

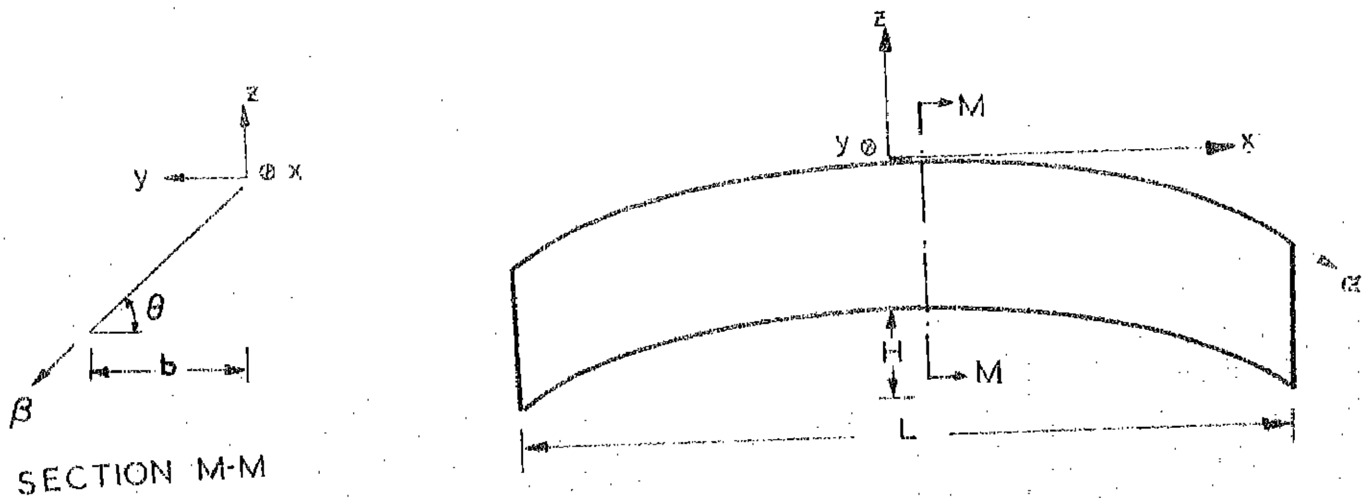


FIG. 1

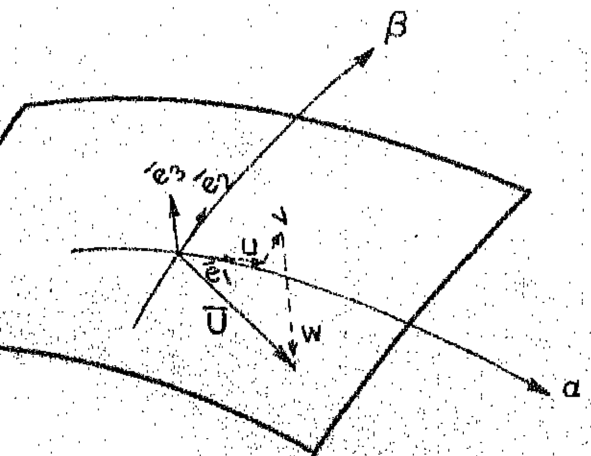


FIG. 2

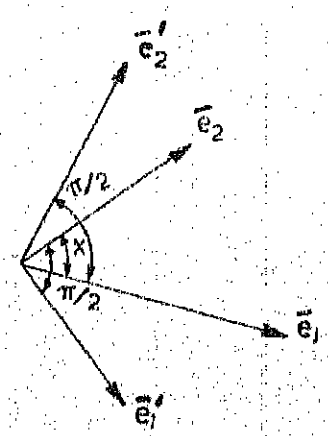


FIG. 3

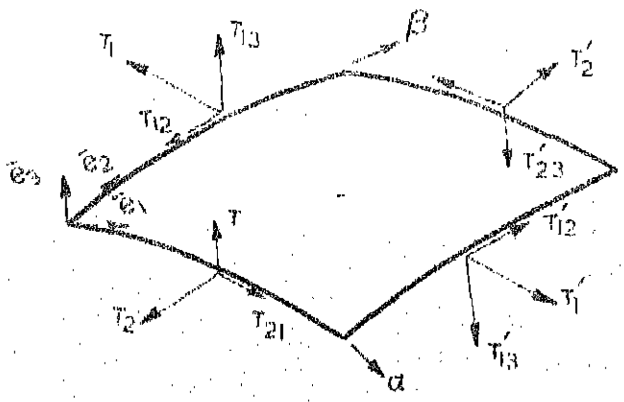


FIG. 4

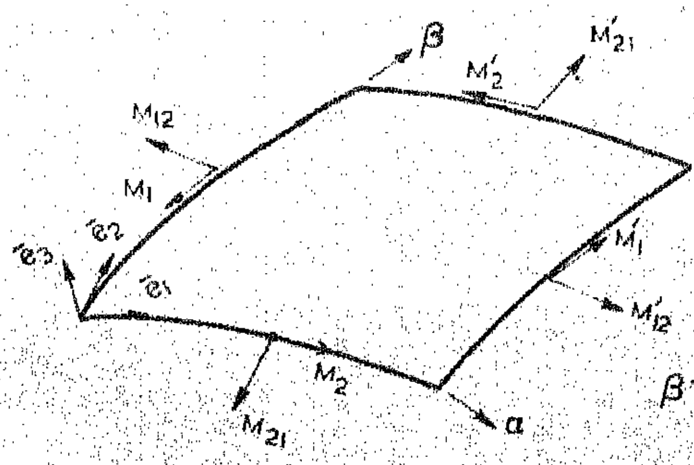


FIG. 5

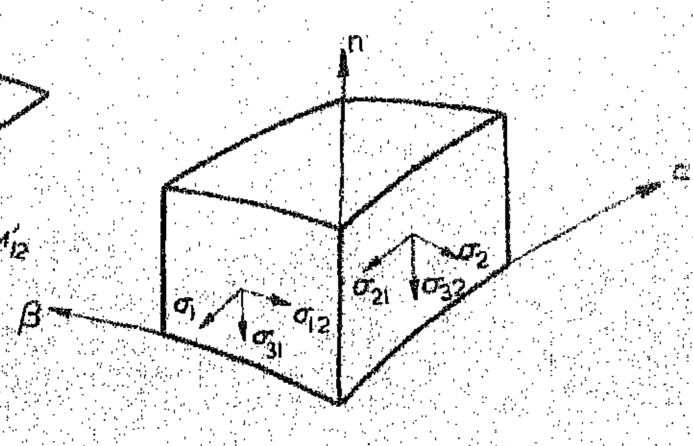


FIG. 6

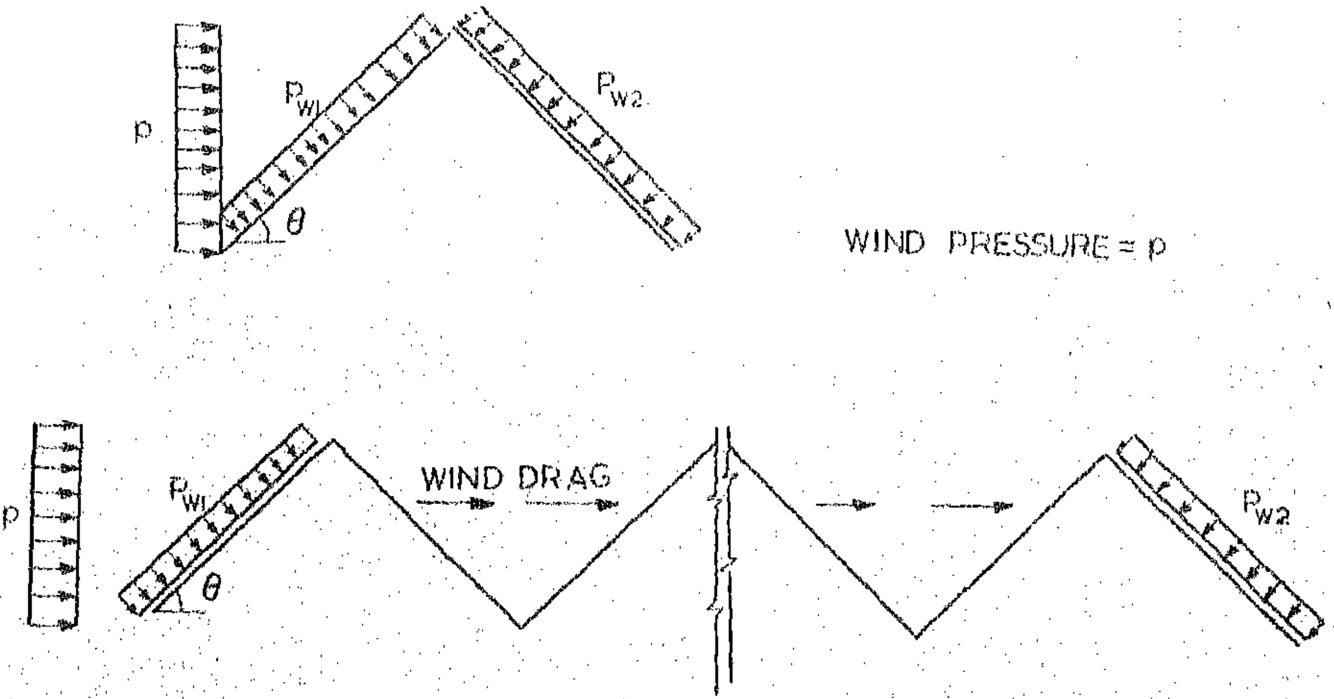


FIG. 7

TABLE I

θ	P_{W1}	P_{W2}	WIND DRAG
20°	−0·40 p	−0·50 p	
30°	−0·10 p	−0·50 p	0·05 p
40°	+0·10 p	−0·50 p	
50°	+0·30 p	−0·50 p	
60°	+0·40 p	−0·50 p	

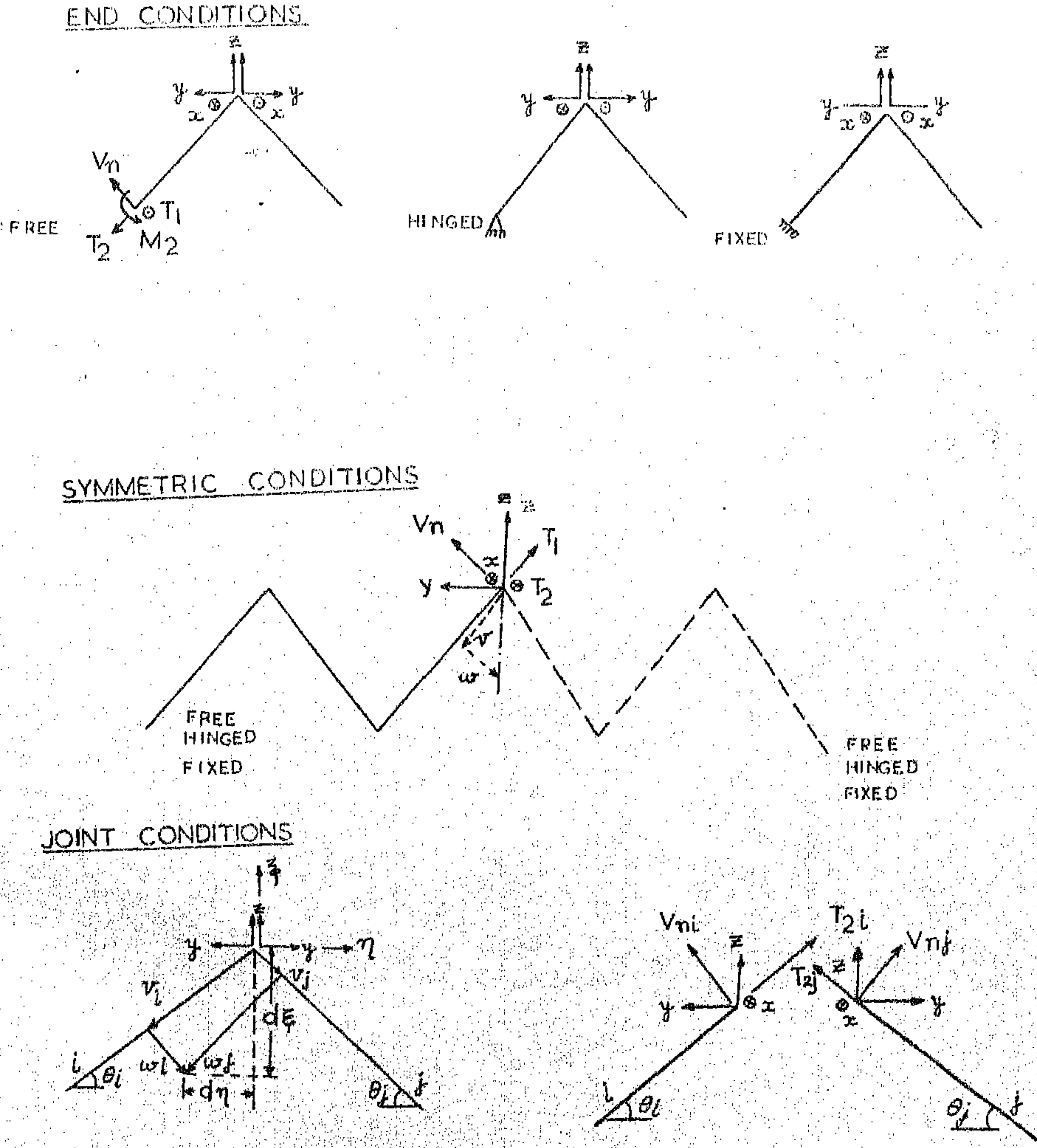

END CONDITIONS
z
y
x
Vn
T1
T2
M2
FREE
HINGED
FIXED
SYMMETRIC CONDITIONS
Vn
T1
T2
y
x
v
w
FREE
HINGED
FIXED
FREE
HINGED
FIXED
JOINT CONDITIONS
ξ
η
dξ
dη
vi
vj
wi
wj
i
j
θi
θj
Vni
T2i
Vnj
T2j

FIG. 8

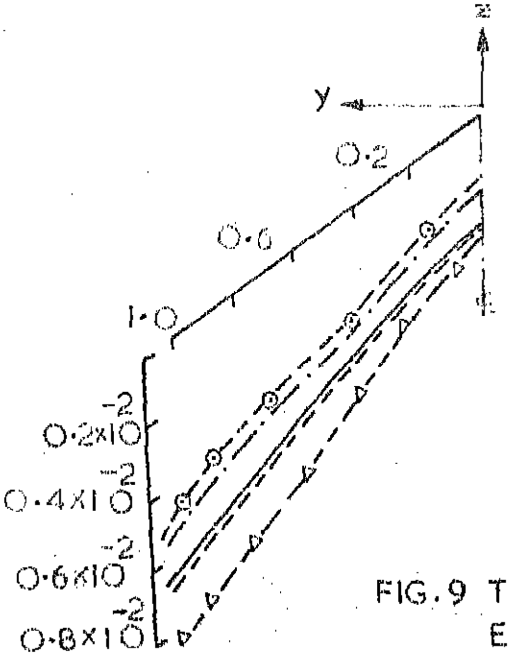


FIG. 9 TRANSVERSE BOUNDARIES FIXED, LONGITUDINAL EDGES FREE, LOAD P = 1000

VERTICAL DEFLECTION

——— $\frac{b}{L}=\frac{1}{7}$, $\frac{H}{L}=\frac{1}{7}$, $\theta=42^\circ$, $\frac{h}{L}=0{\cdot}00625$, $\nu=0{\cdot}15$

—·— $\frac{b}{L}=\frac{1}{7}$, $\frac{H}{L}=\frac{1}{7}$, $\theta=36^\circ$, $\frac{h}{L}=0{\cdot}005$, $\nu=0{\cdot}15$

– – – $\frac{b}{L}=\frac{1}{7}$, $\frac{H}{L}=\frac{1}{7}$, $\theta=36^\circ$, $\frac{h}{L}=0{\cdot}00625$, $\nu=0{\cdot}15$

—Δ— $\frac{b}{L}=\frac{1}{7}$, $\frac{H}{L}=\frac{1}{8}$, $\theta=36^\circ$, $\frac{h}{L}=0{\cdot}00625$, $\nu=0{\cdot}15$

—o— $\frac{b}{L}=\frac{1}{7}$, $\frac{H}{L}=\frac{1}{8}$, $\theta=42^\circ$, $\frac{h}{L}=0{\cdot}005$, $\nu=0{\cdot}15$.

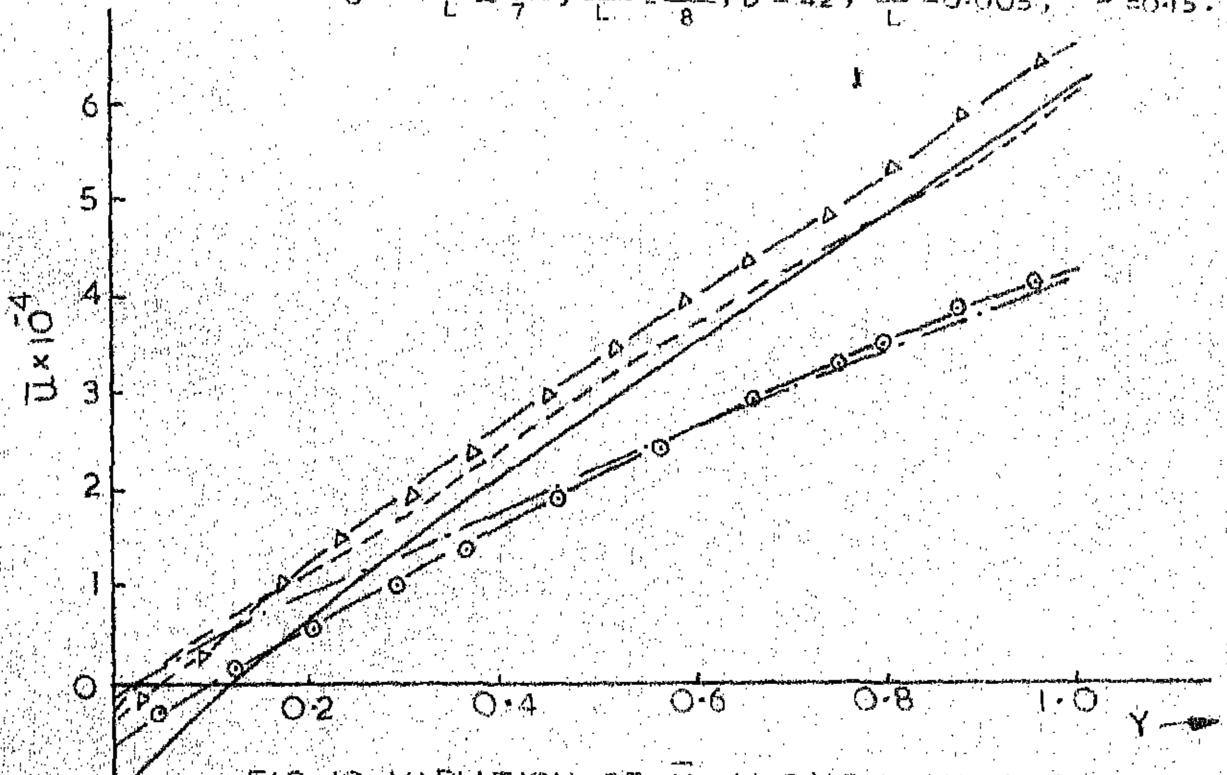


FIG. 10. VARIATION OF $\bar{U}$ ALONG Y AT X=0

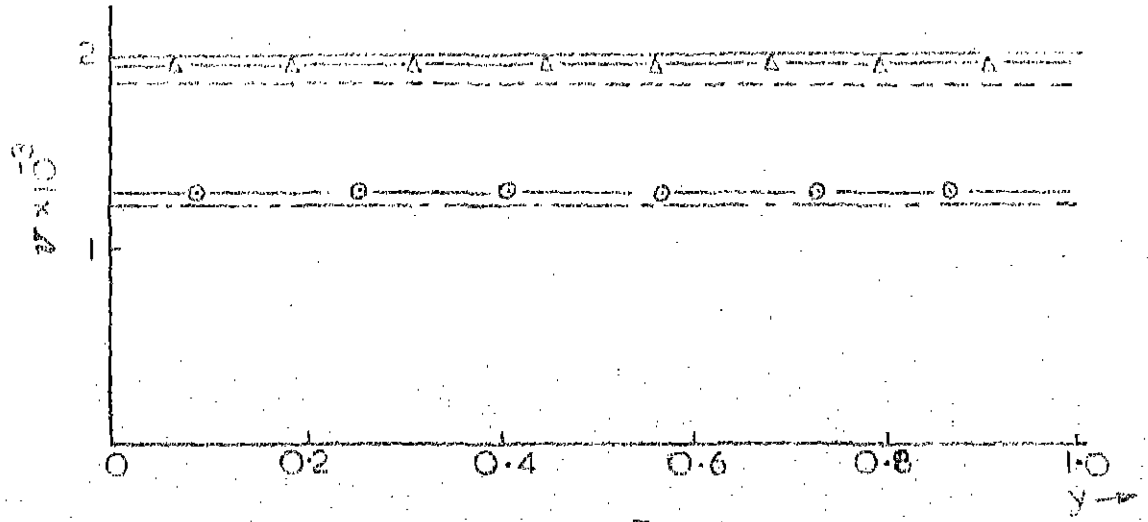


FIG. 11 VARIATION OF $\bar{V}$ ALONG Y AT X=O

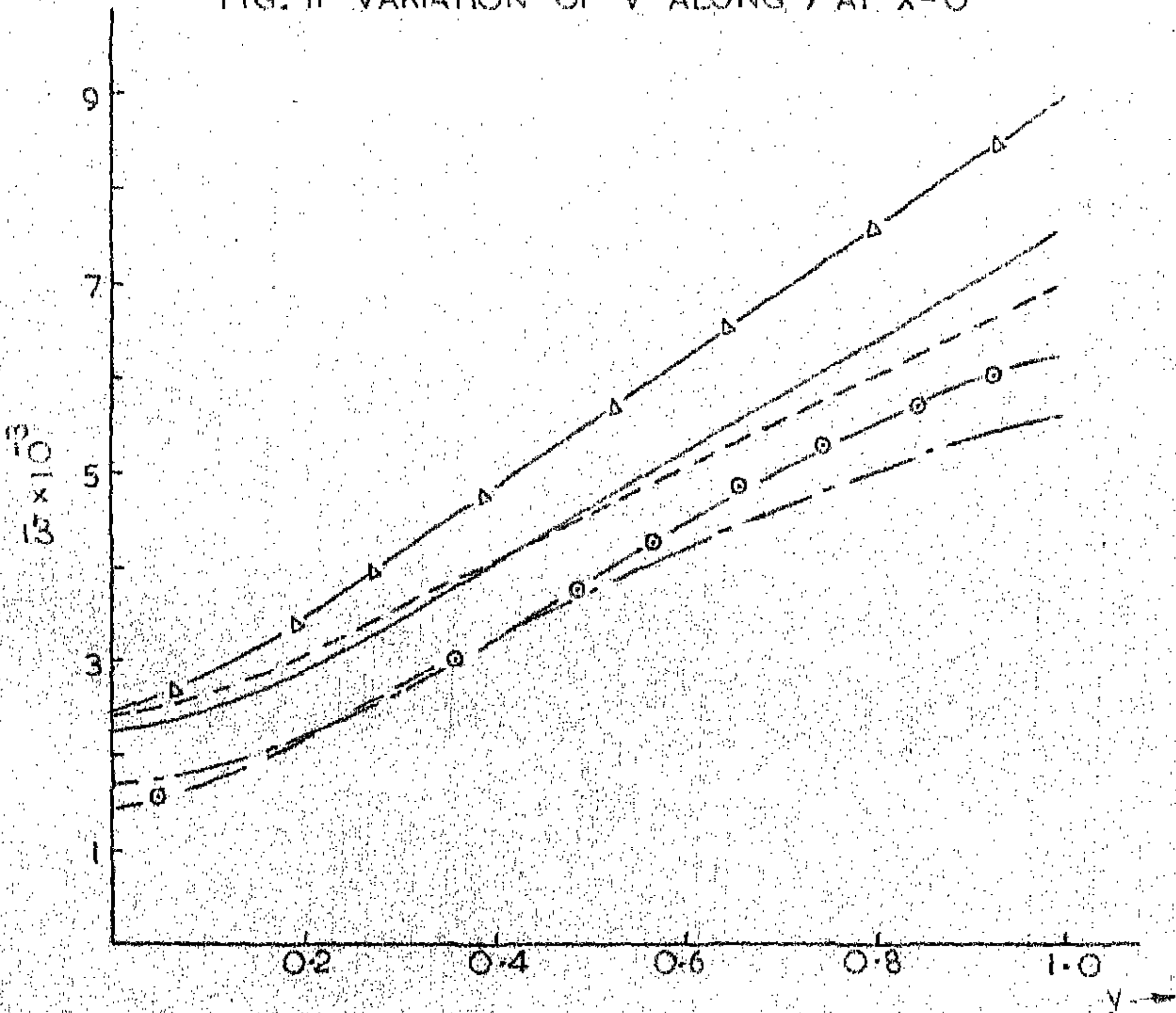


FIG.12 VARIATION OF $\bar{W}$ ALONG Y AT X=O

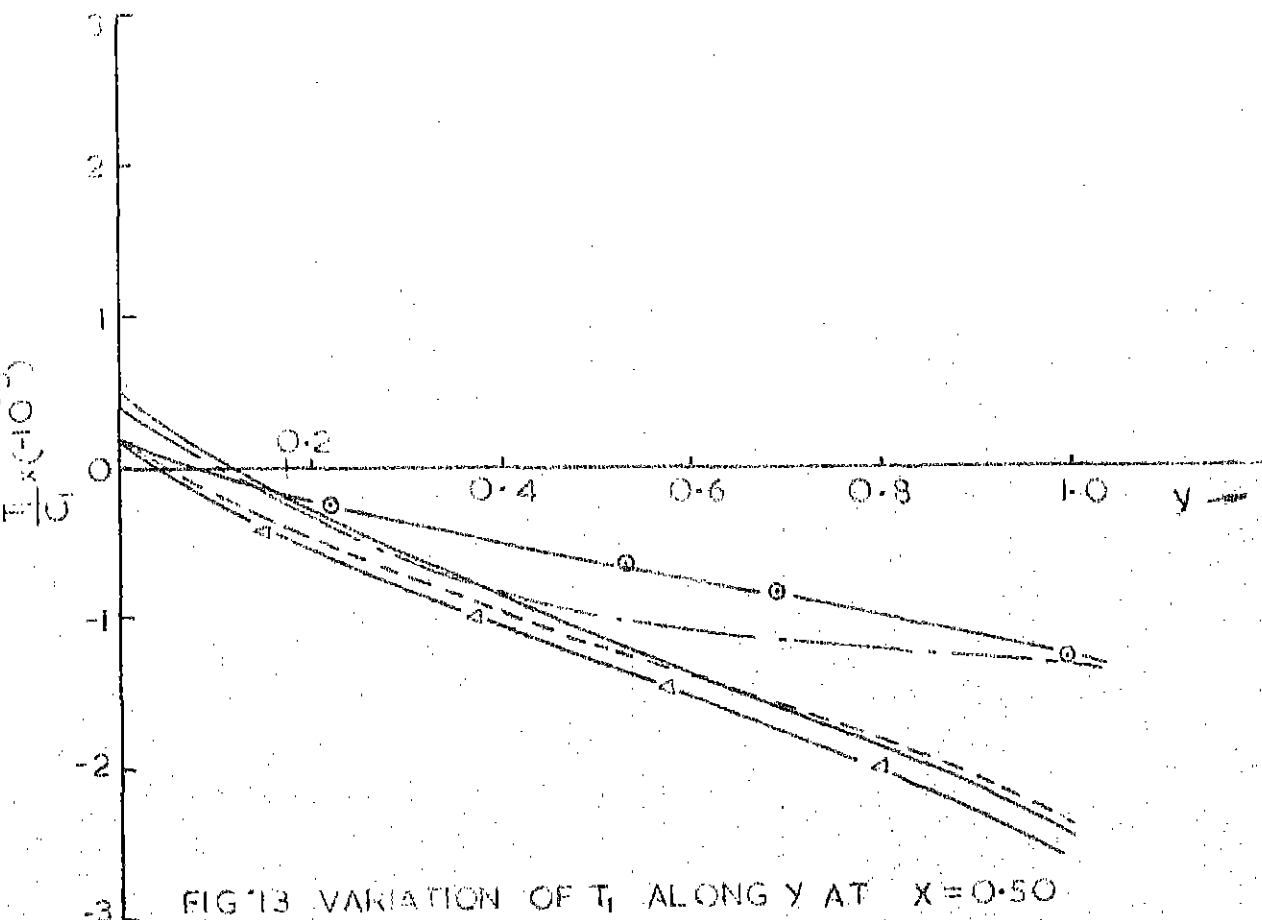
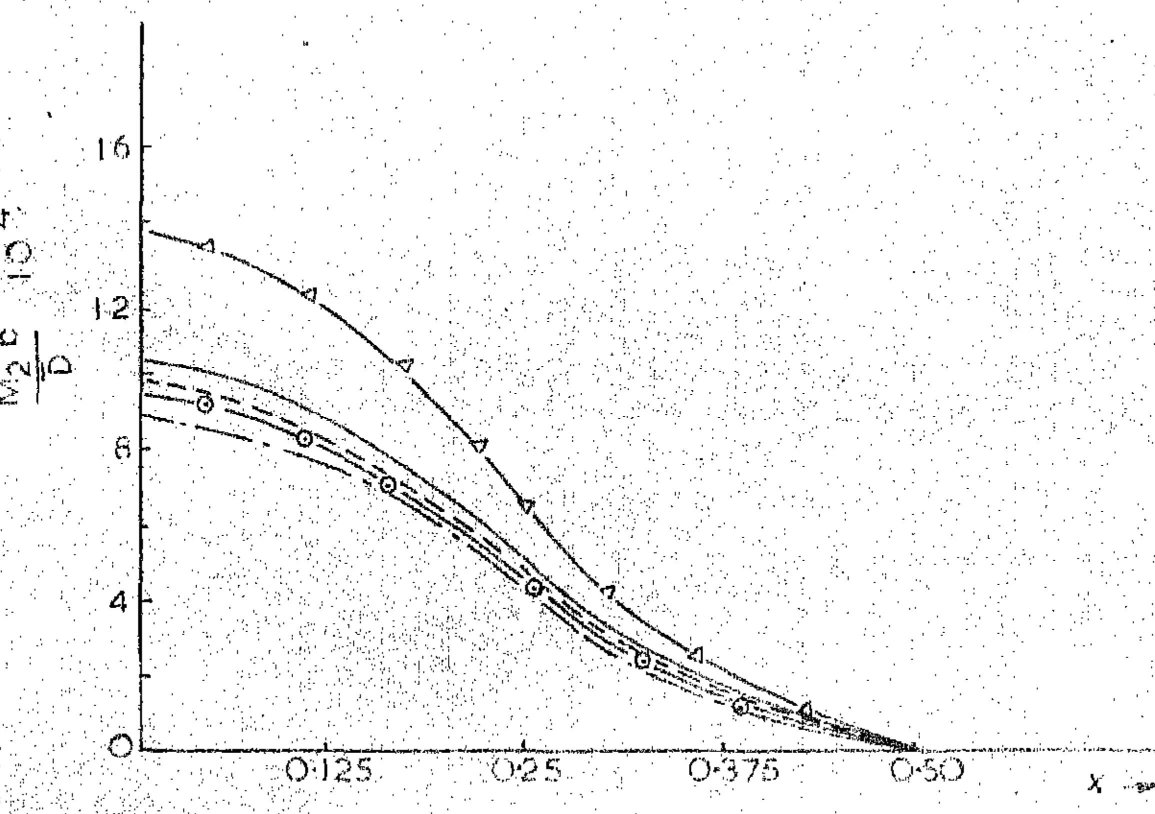
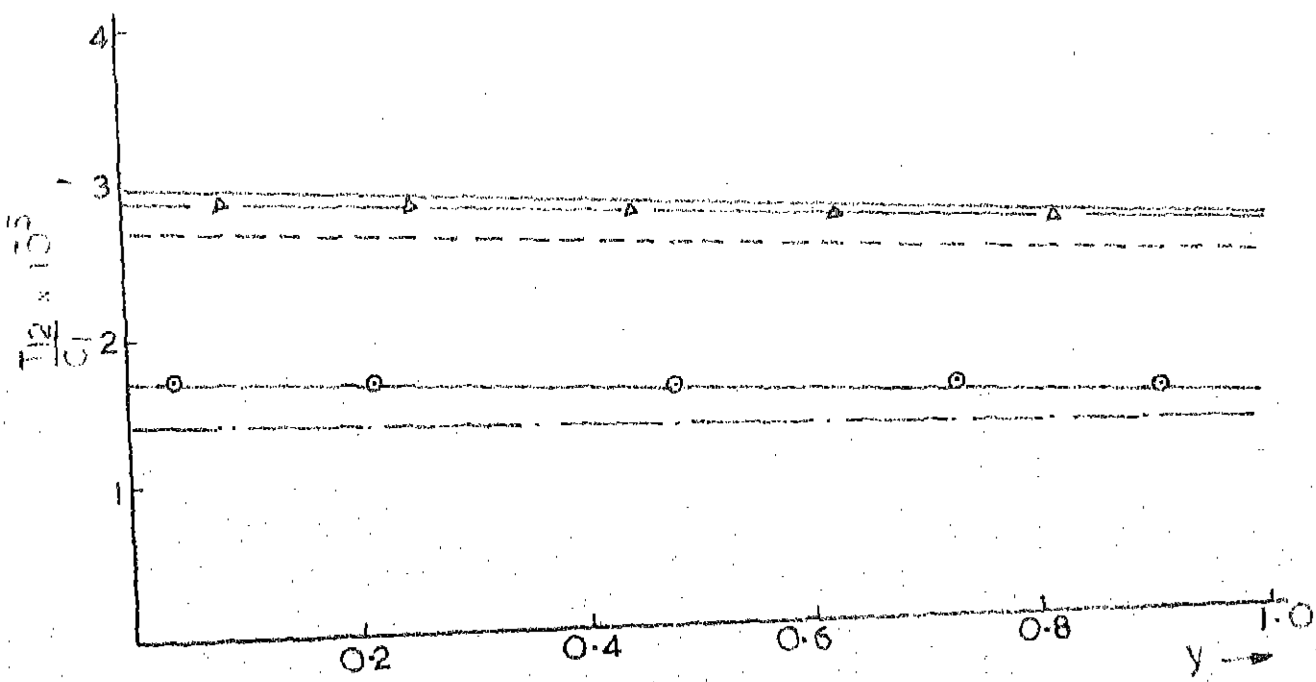


FIG 13 VARIATION OF T_1 ALONG Y AT X = 0·50





DIG. 15 DISTRIBUTION OF T_{12} ALONG y AT X=0·50

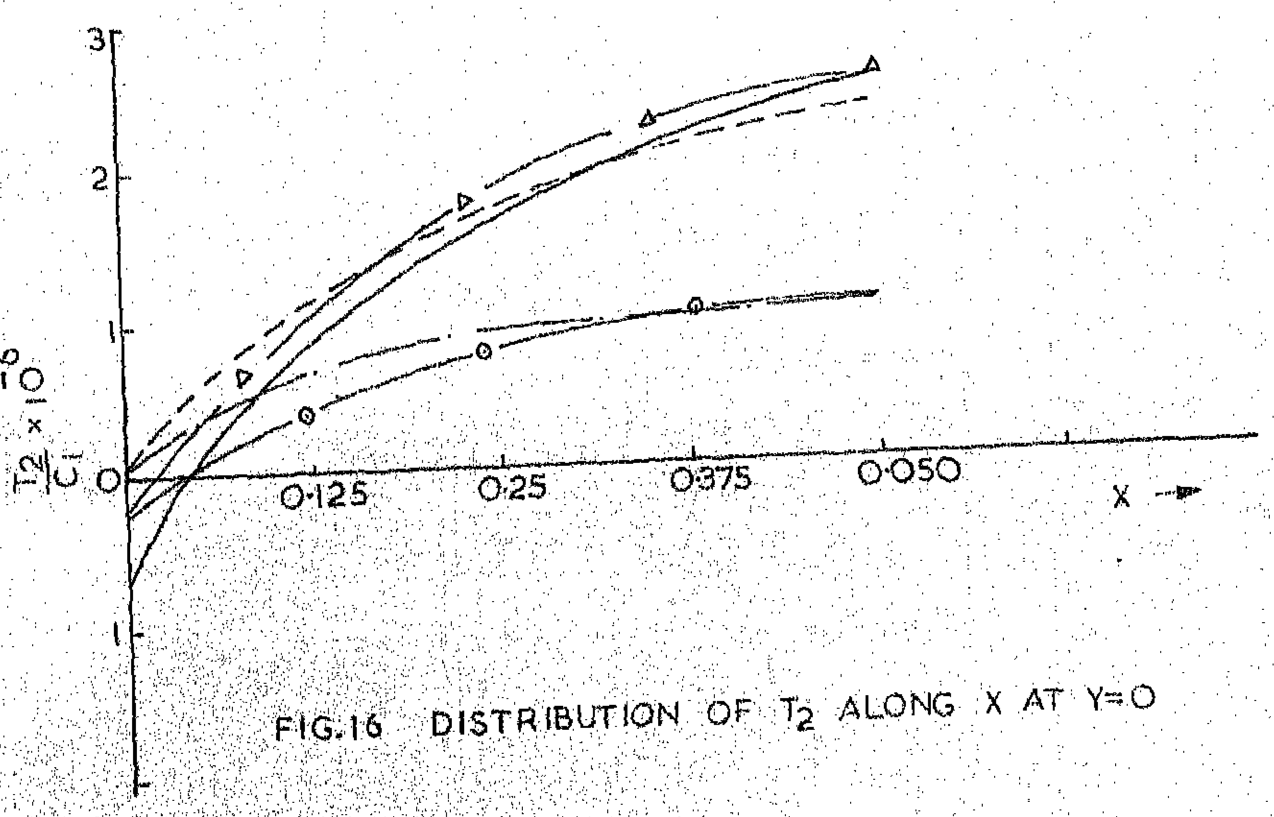


FIG.16 DISTRIBUTION OF T_2 ALONG X AT Y=O

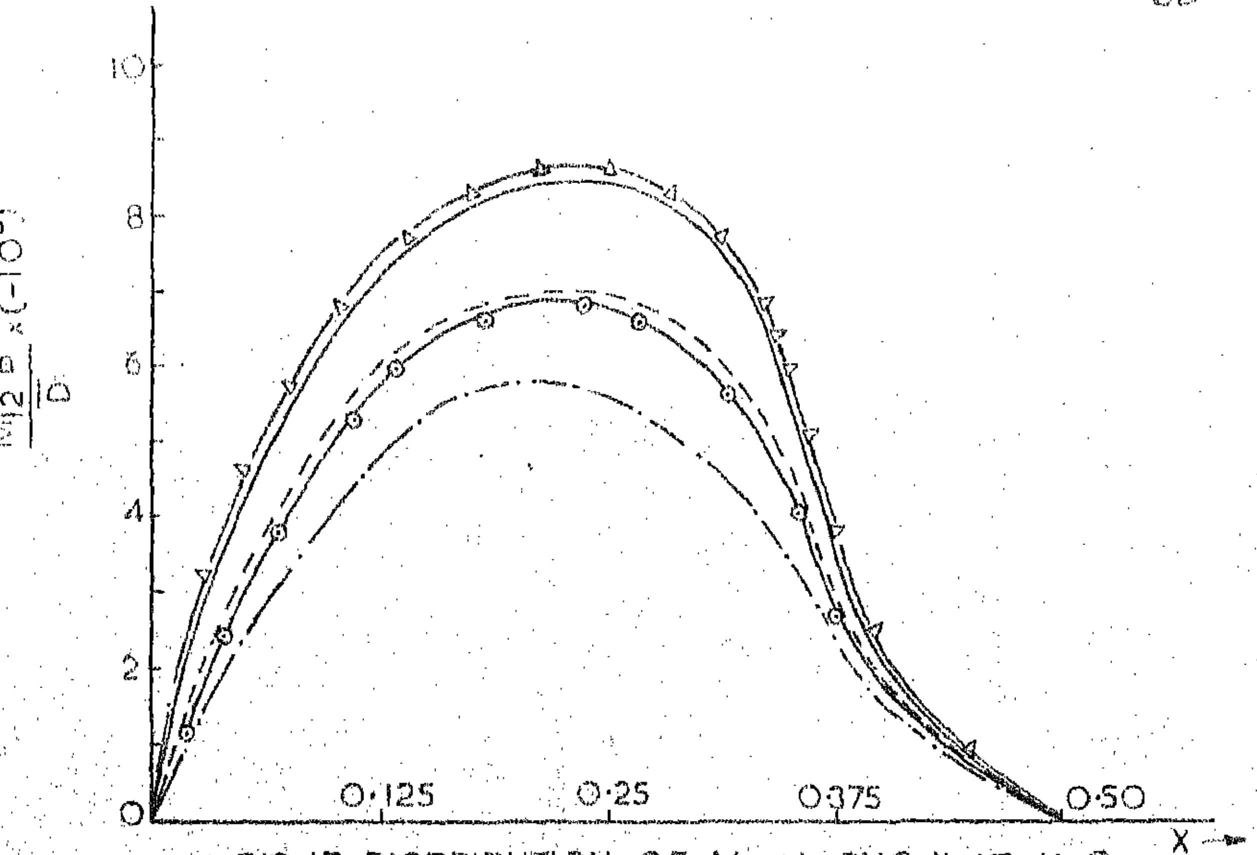


FIG. 17 DISTRIBUTION OF M_{12} ALONG X AT Y=0

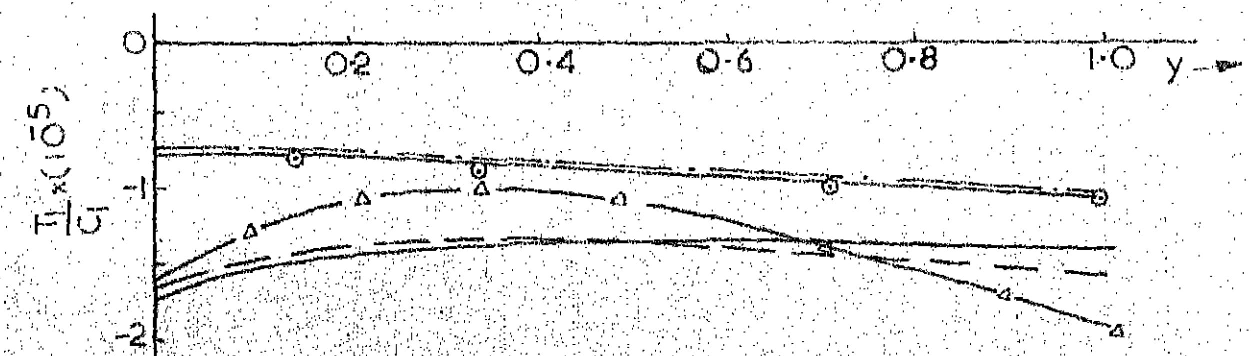
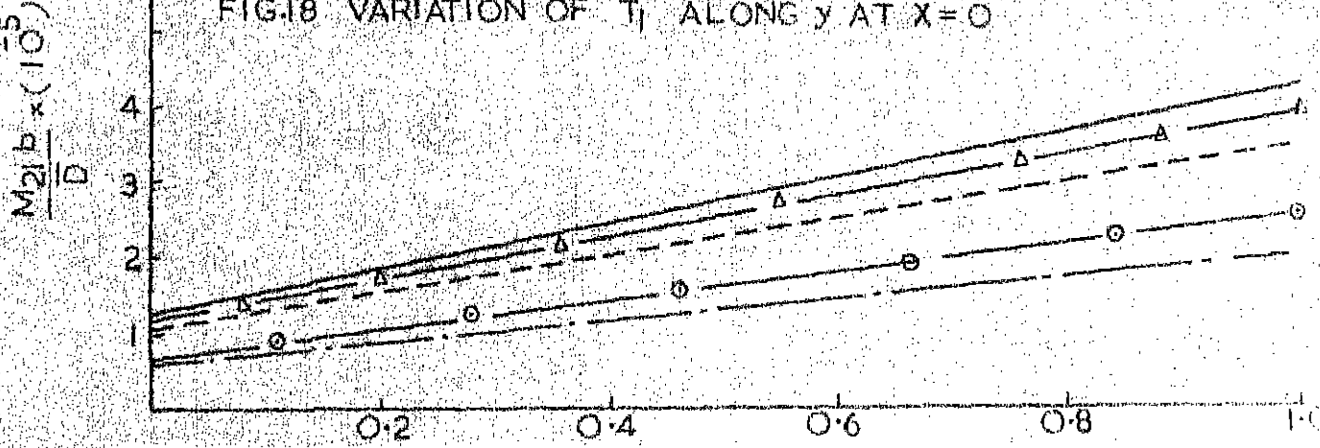


FIG. 18 VARIATION OF T_1 ALONG y AT X=0



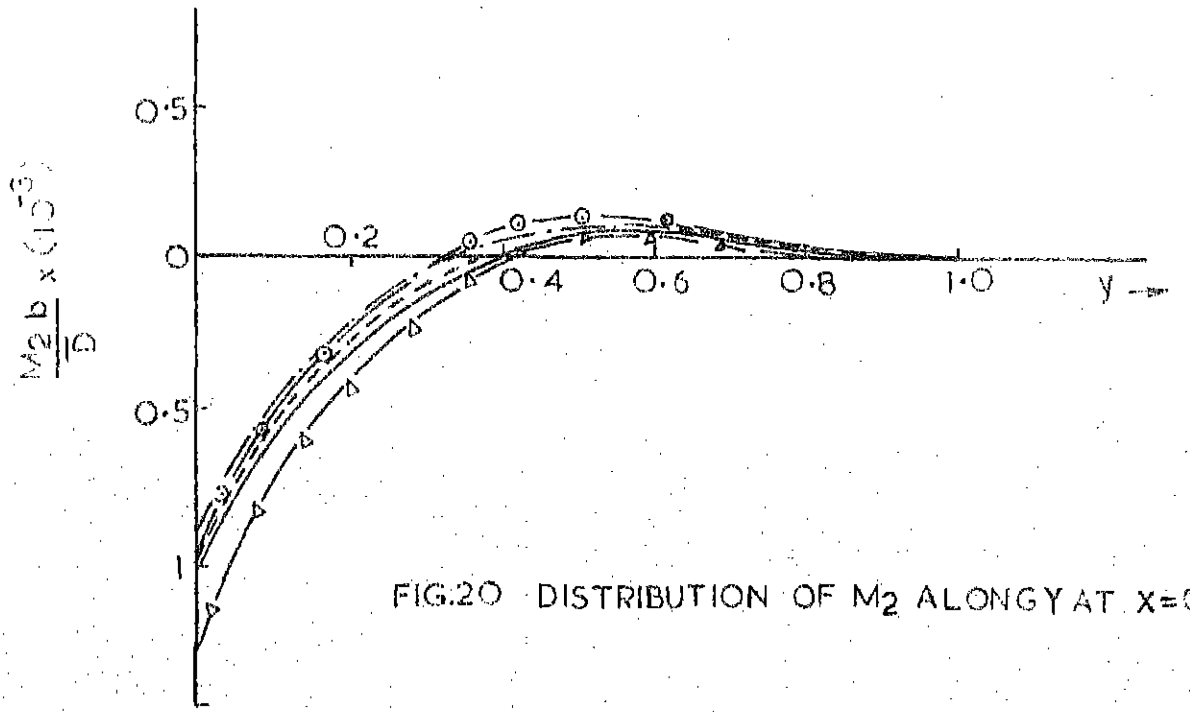


FIG.2O DISTRIBUTION OF M_2 ALONG Y AT X=O

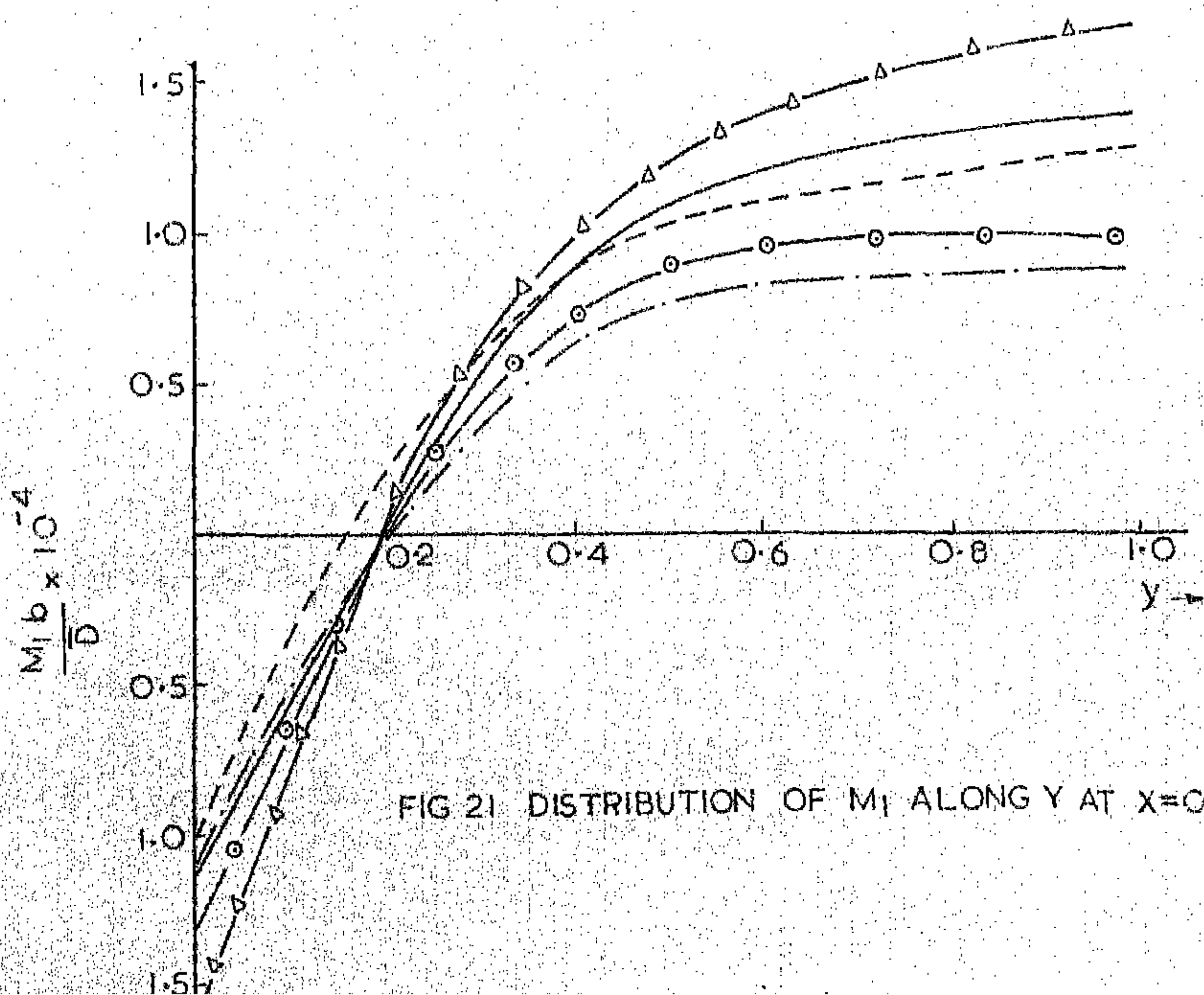


FIG 21 DISTRIBUTION OF M_1 ALONG Y AT X=O

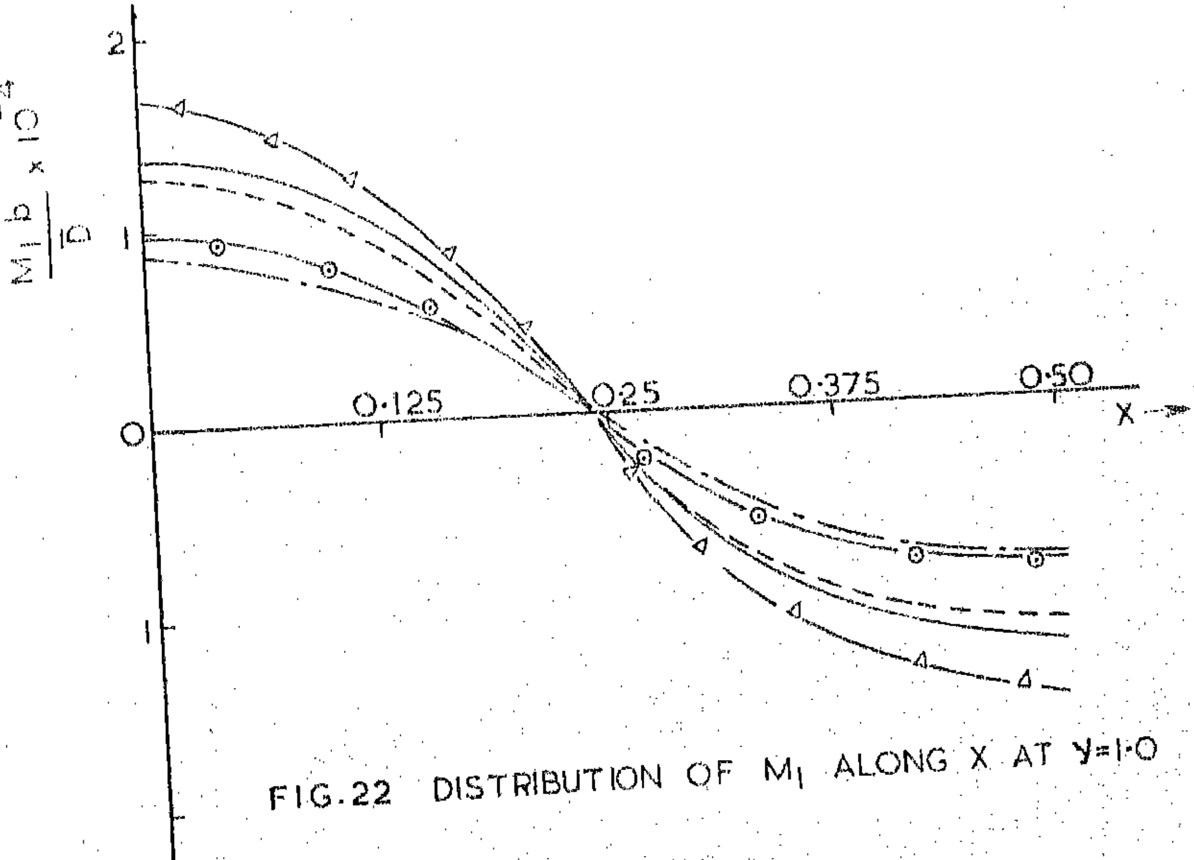


FIG. 22 DISTRIBUTION OF M_1 ALONG X AT $y=1·0$

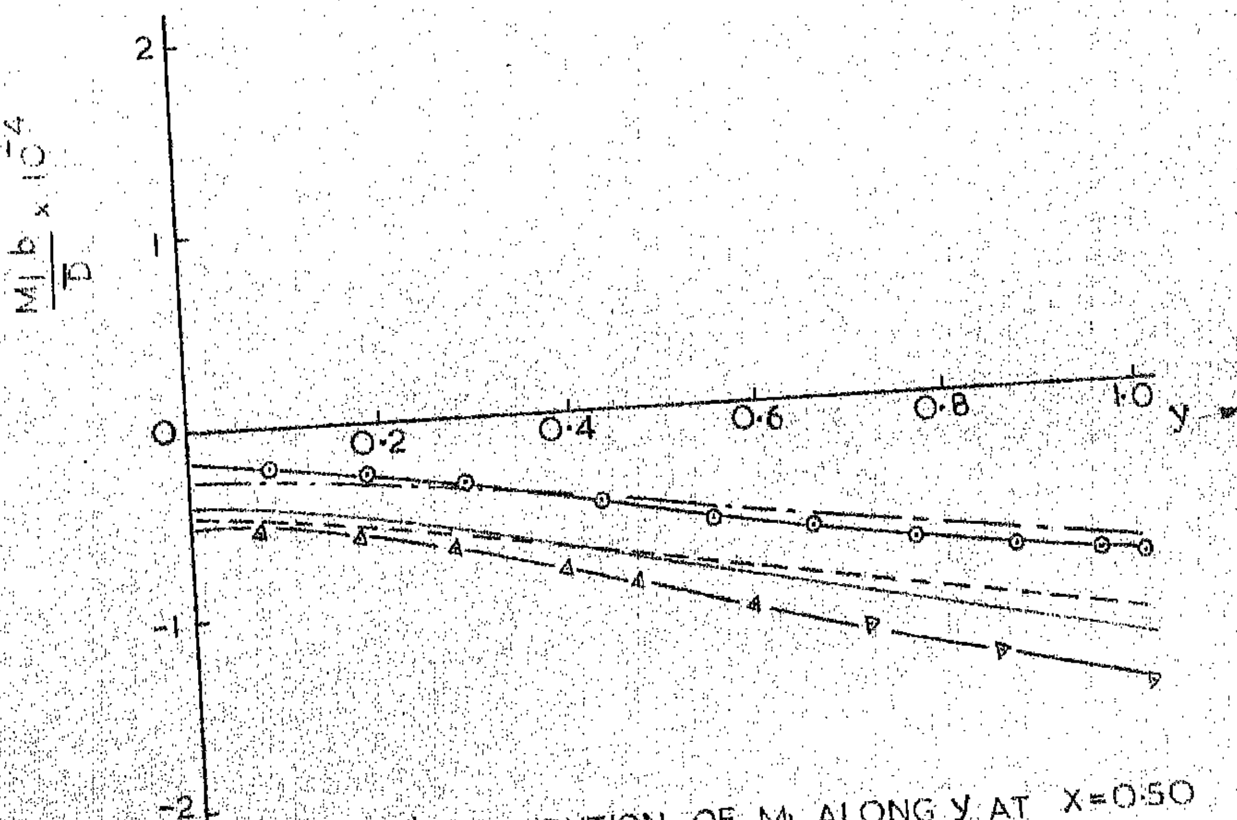
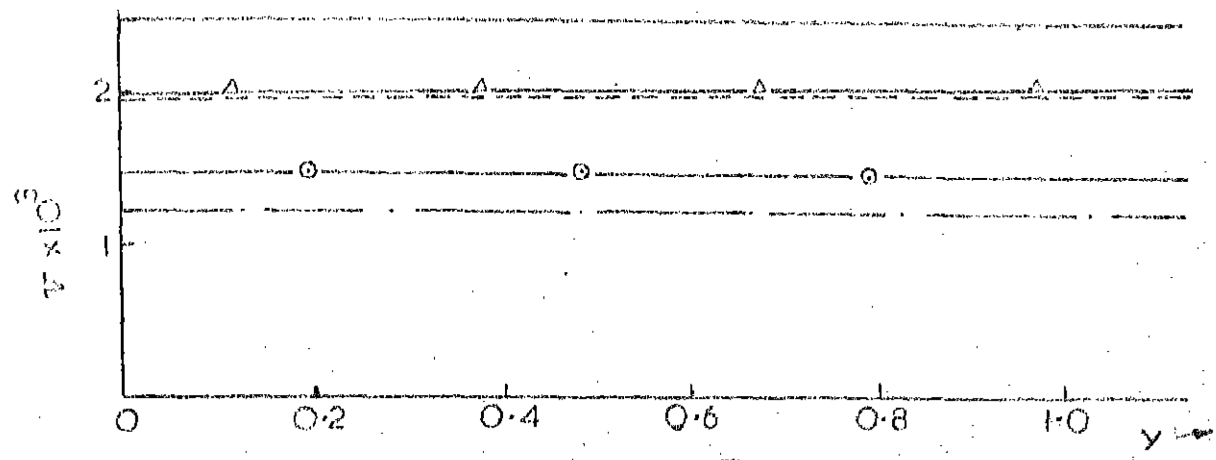


FIG. 23 DISTRIBUTION OF M_1 ALONG y AT $X=0·50$



FIG·26 DISTRIBUTION OF $\bar{V}$ ALONG Y AT X=O

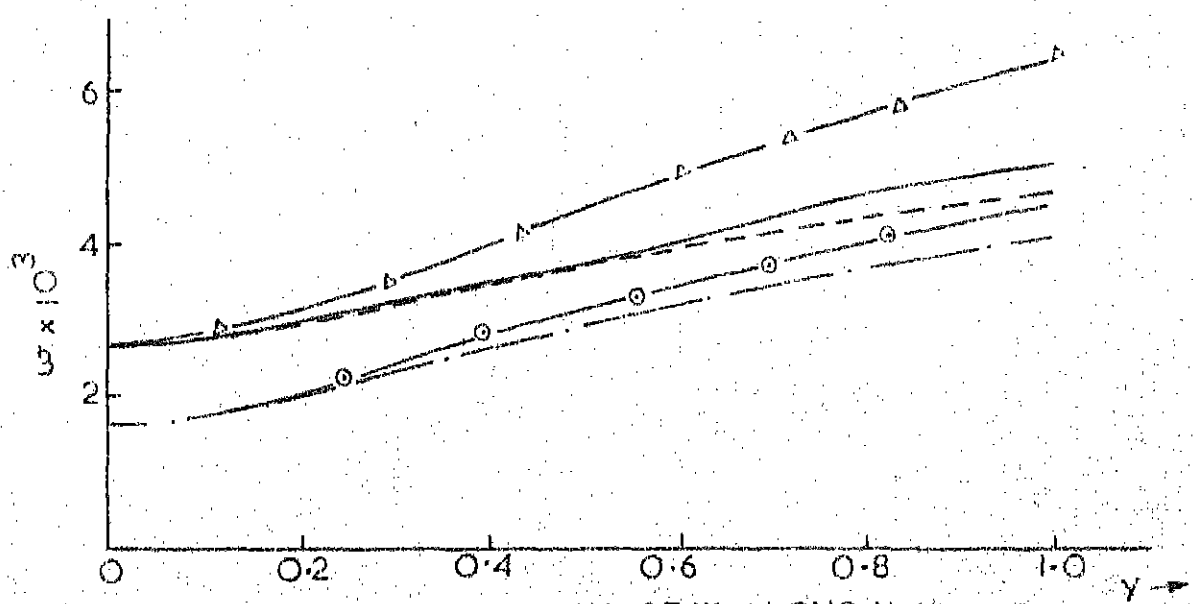


FIG. 27 DISTRIBUTION OF W ALONG Y AT X=O

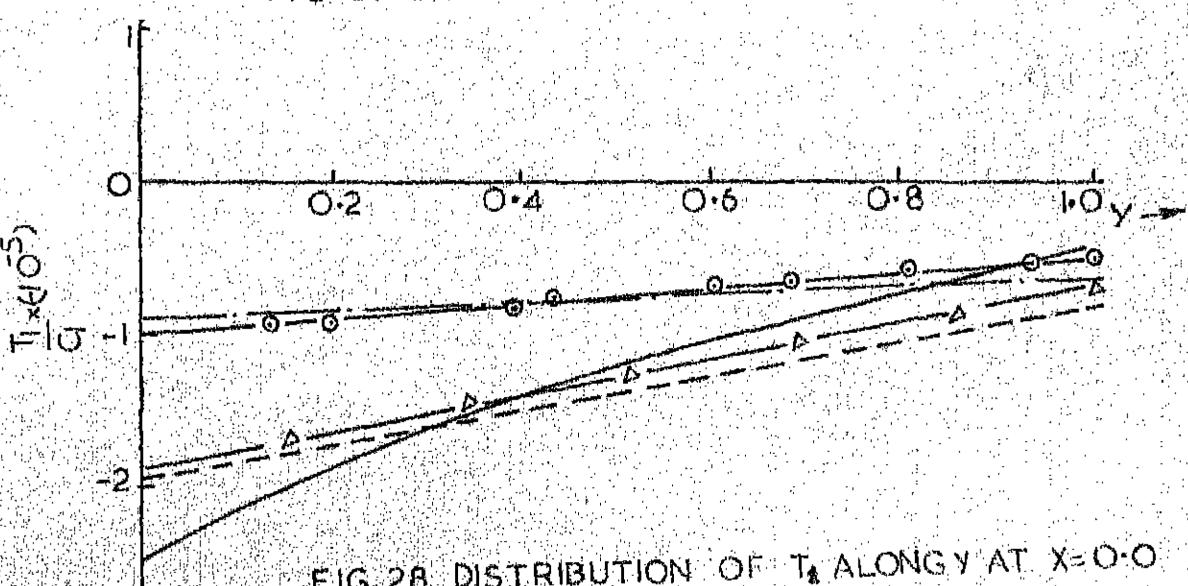


FIG. 28 DISTRIBUTION OF T_1 ALONG Y AT X=O·O

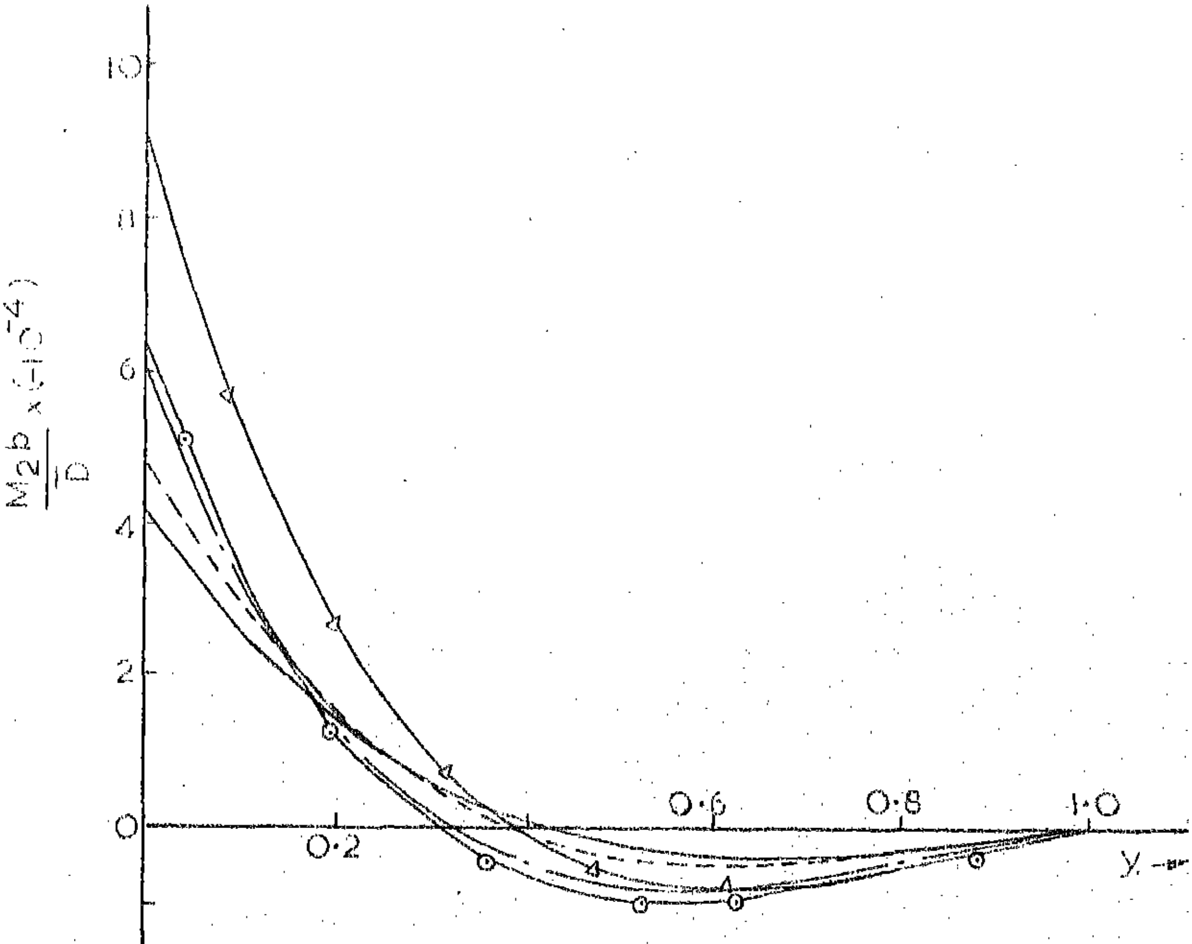
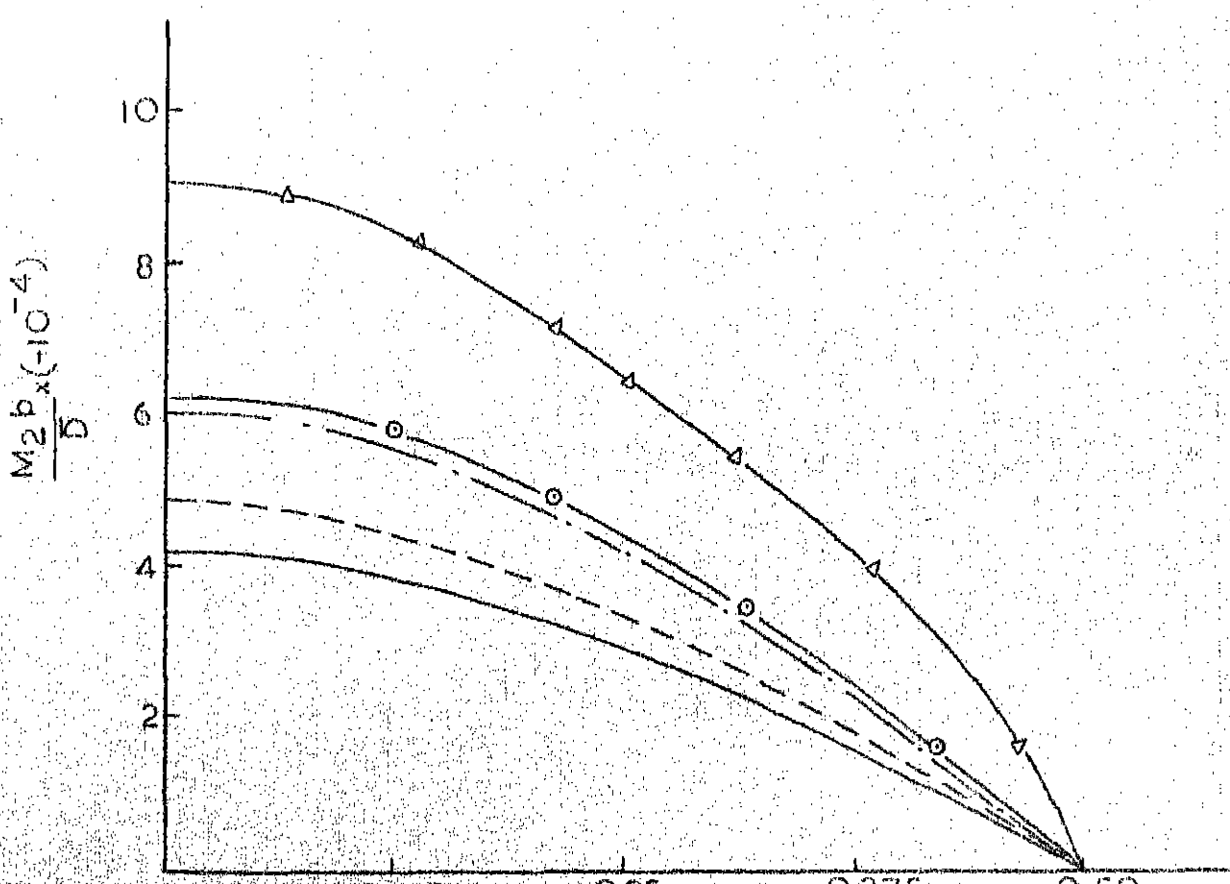


FIG.29 DISTRIBUTION OF M_2 ALONG Y AT X=O



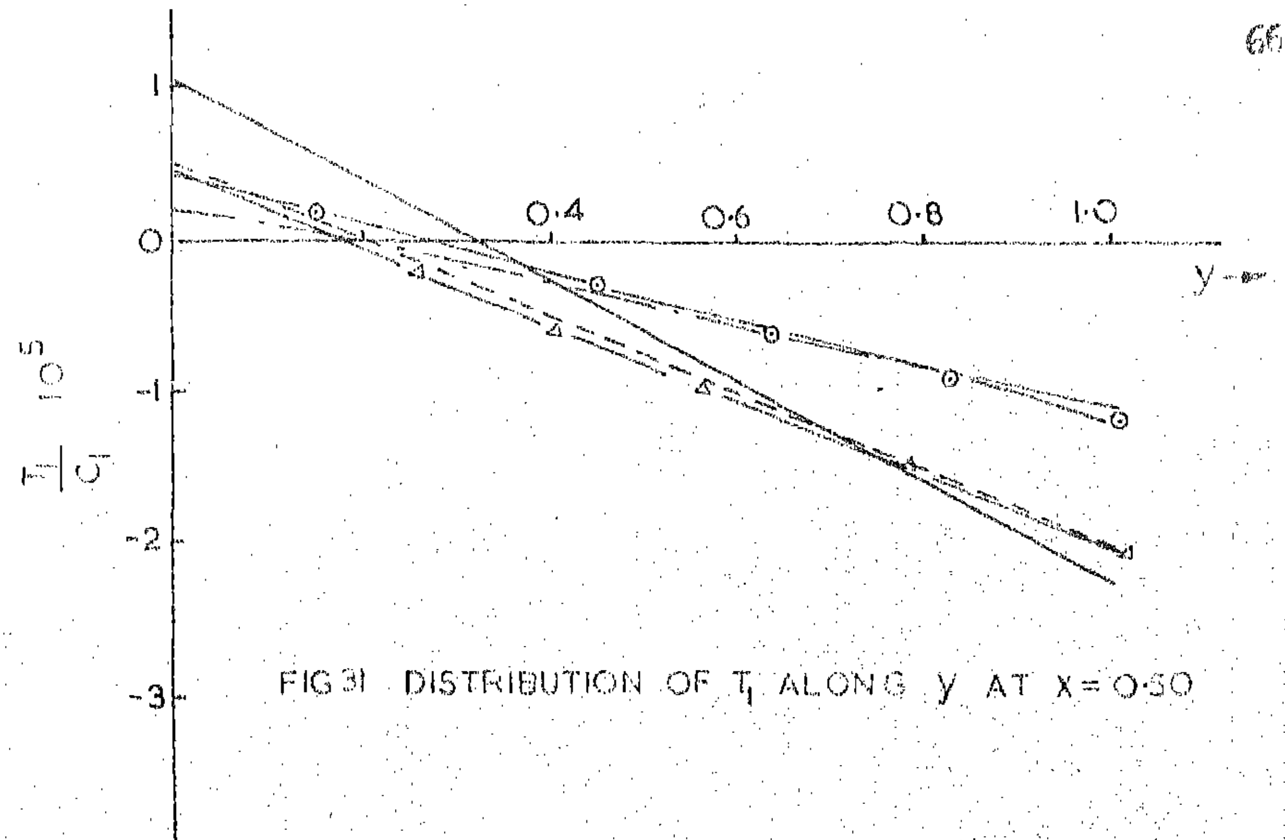


FIG 31 DISTRIBUTION OF T_1 ALONG y AT $X = 0{\cdot}50$

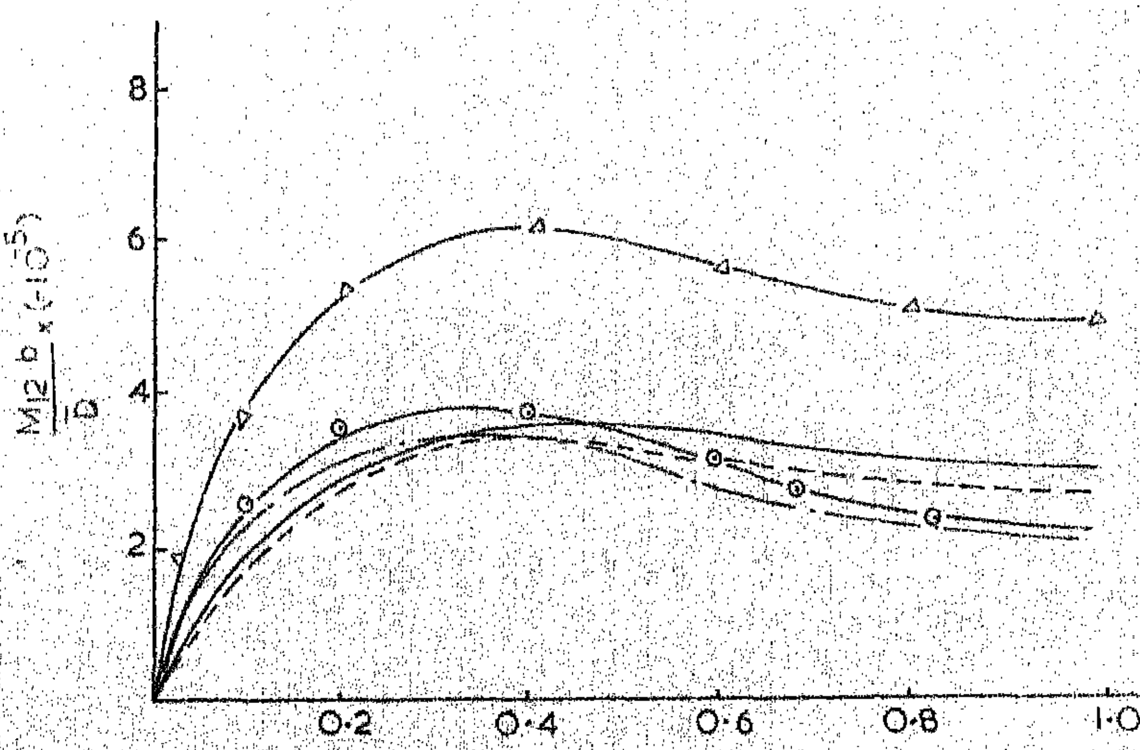


FIG. 32 DISTRIBUTION OF M_{12} ALONG y AT $X = 0{\cdot}50$

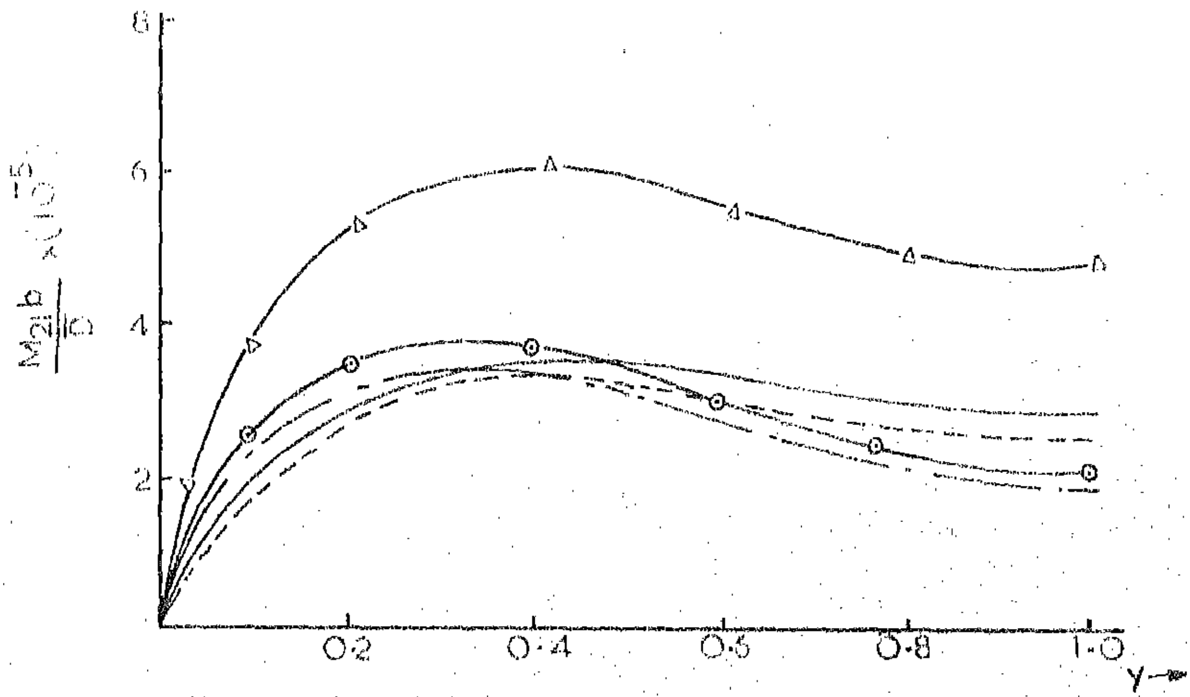


FIG. 33 DISTRIBUTION OF M_{21} ALONG Y AT X = 0·50

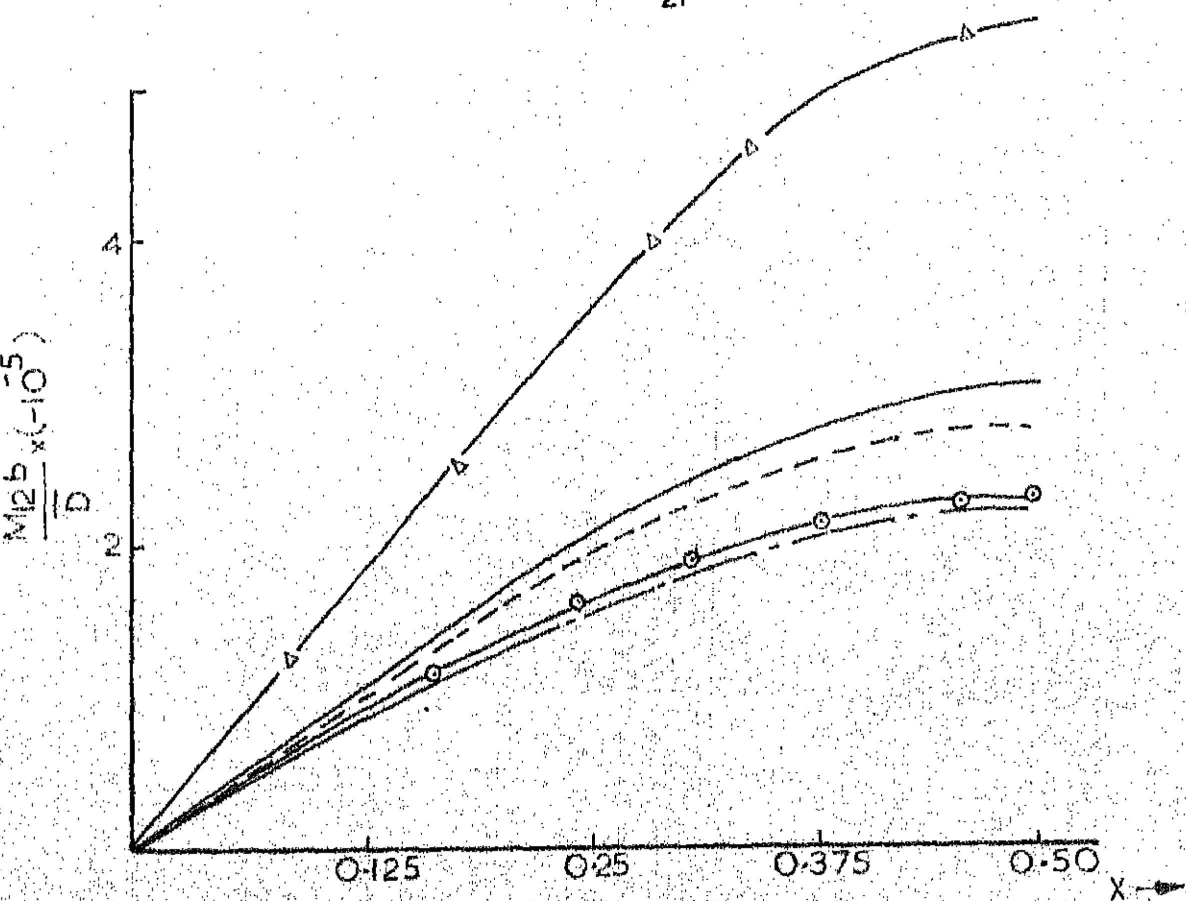


FIG. 34 DISTRIBUTION OF M_{12} ALONG X AT Y = 1·0

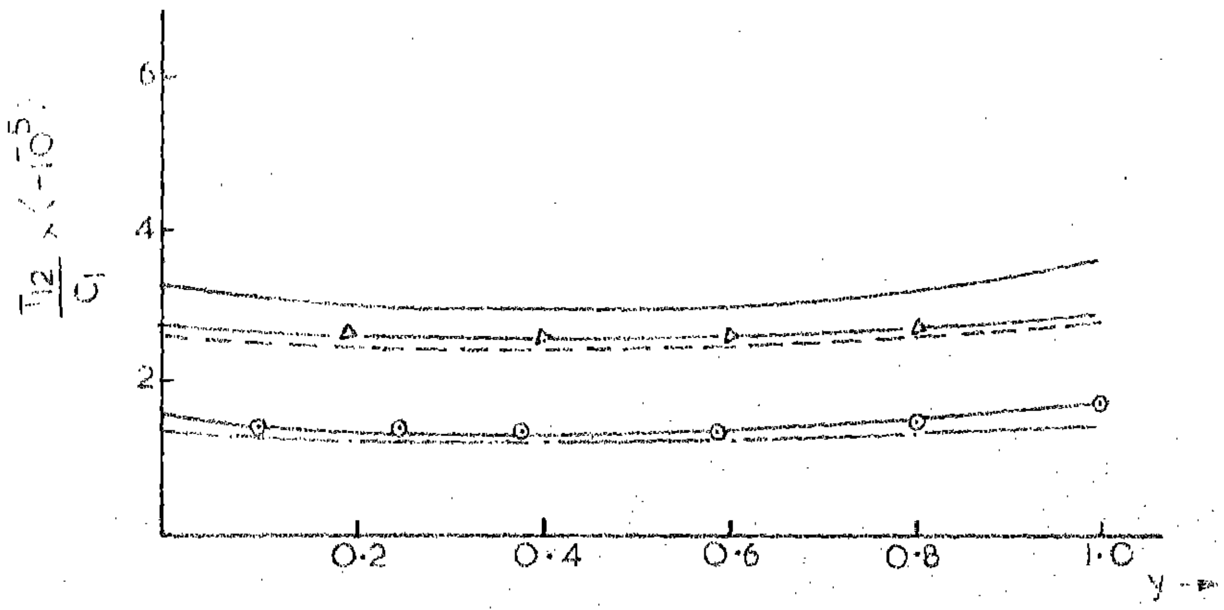


FIG. 35 DISTRIBUTION OF T_{12} ALONG Y AT X = 0·50

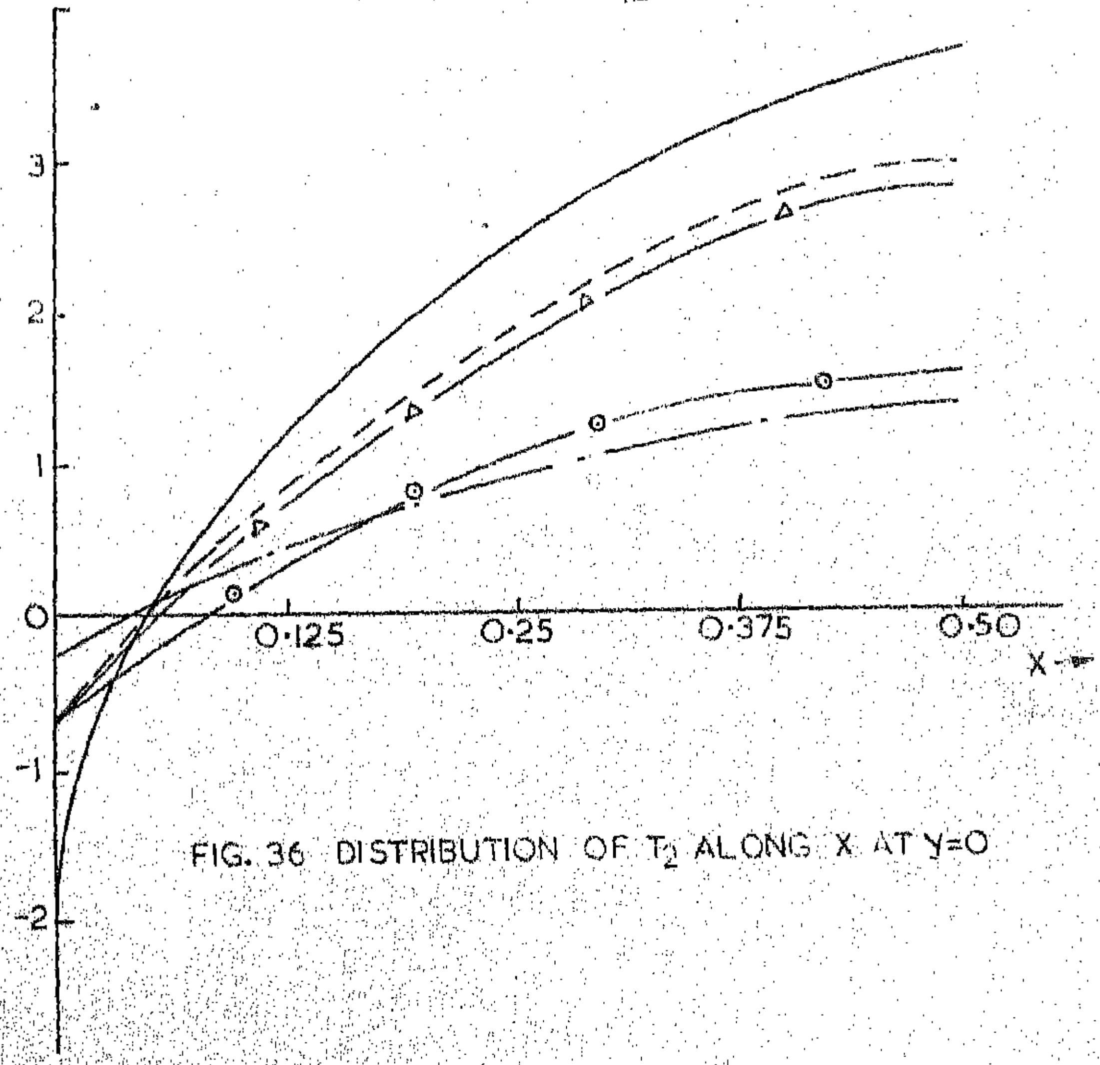


FIG. 36 DISTRIBUTION OF T_2 ALONG X AT Y=0

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APPENDIX A

$$t = \tan\theta$$

$$Z_x = -8a_1x$$

$$Z_y = -t$$

$$Z_{xx} = -8a_1$$

$$a_4 = \sqrt{1 + t^2}$$

$$a_5 = a_1t(1 - \upsilon)$$

$$a_6 = a_1^2t^2(1 + \upsilon)$$

$$a_7 = a_2^2a_3^2$$

$$a_8 = a_2a_3^2$$

$$\frac{1}{R_1} = \frac{8a_1}{a_4L}$$

$$A_1 = -192\, a_6/a_3^2$$

$$A_2 = -48\, a_5/a_8$$

$$A_3 = 6a_4^2/a_3^2$$

$$A_4 = -32\, A_3a_6/a_4^2$$

$$A_5 = 3(1 - \upsilon)/a_7$$

$$A_6 = -48\ a_5/a_8$$

$$A_7 = 24\ a_5 a_4/a_3^2$$

$$A_8 = -192\ a_6/a_4 a_7$$

$$A_9 = 24\ a_5/a_4 a_7$$

$$A_{10} = -192\ a_1^2 t^2 (3 - \nu)/a_4 a_8$$

$$A_{11} = 3a_4(1 + \nu)/a_8$$

$$A_{12} = -48\ a_1 a_4/a_3^2$$

$$A_{13} = 384\ a_1^2 t(1 - \nu)/a_4 a_8$$

$$A_{14} = 4\ a_1 a_4^2$$

$$B_1 = 8\ a_5/a_3^2$$

$$B_2 = -128\ t^2 a_1^2 (1 - \nu)/a_4^2 a_8$$

$$B_3 = B_1$$

$$B_4 = B_1\ (a_2 a_4)^2$$

$$B_5 = -64\ a_1^2 t^2 (3 - \nu)/a_8 a_4^2$$

$$B_6 = (1 + \nu)/a_8$$

$$B_7 = a_4(1 - \nu)/a_3^2$$

$$B_8 = -B_1/a_2 a_4$$

$$B_9 = 2/a_4 a_7$$

$$B_{10} = -64\ a_6/a_4^3 a_7$$

$$B_{11} = -16\ a_5/a_4 a_8$$

$$B_{12} = 1024\ a_1^3 t^2/a_4^3 a_8$$

$$B_{13} = t/6$$

$$B_{14} = 64\ B_{13}\ a_1^2/a_4^2$$

$$B_{15} = -16\ a_1\ /a_4 a_8$$

$$C_1 = -96\ a_1/a_3^2$$

$$C_2 = 768\ a_1^2/a_4 a_3^2$$

$$C_3 = C_2 t(1 - \nu)/a_2 a_4$$

$$C_4 = C_1\nu/a_2a_4$$

$$C_5 = -48\ a_1t/a_2a_4$$

$$C_6 = 512\ a_1^2t^2/a_2^2a_4^3$$

$$C_7 = a_4$$

$$C_8 = -2C_5/3$$

$$C_9 = C_8/(a_2a_4)^2$$

$$C_{10} = 64\ a_1^2t^2(2-\nu)/a_2^2a_4^3$$

$$C_{11} = C_9/2$$

$$C_{12} = 1/(a_2^4a_4^3)$$

$$C_{13} = 2/a_2^2a_4$$

$$C_{14} = 128\ C_{13}a_1^2t^2/a_4^2$$

$$C_{15} = -1$$

(a) Fixed Supports:

$$\bar{A}_1 = -A_1/4 - 2\pi^2A_3 + A_4\left(\frac{1}{4} - \frac{\pi^2}{6}\right)$$

$$\bar{A}_2 = A_2/2 - A_6/4$$

$$\bar{A}_3 = A_5/2$$

$$\bar{A}_4 = -8\ A_7/9$$

$$\bar{A}_5 = 8\ A_8/9\pi^2 + A_{10}\left(\frac{57}{27\pi^2} - \frac{1}{3}\right) - A_{11}\cdot\frac{4}{3}$$

$$\bar{A}_6 = -\pi A_{12}$$

$$\bar{A}_8 = A_{13}\cdot\frac{3}{8\pi}$$

$$\bar{A}_9 = A_{14}/2\pi$$

$$\bar{B}_1 = 4\ B_1/3 - 32\ B_3/9$$

$$\bar{B}_2 = 8\ B_2/9\pi^2 + B_5\left(\frac{1}{3} - \frac{104}{27\pi^2}\right) + 4\ B_6/3$$

$$\bar{B}_3 = 8\ B_4/9\pi^2$$

$$\bar{B}_4 = -B_7\cdot\pi^2/2$$

$$\bar{B}_5 = B_8/2 - B_{11}/4$$

$$\bar{B}_6 = B_9/2 + B_{10}\left(\frac{1}{24} - \frac{1}{4\pi^2}\right)$$

$$\bar{B}_7 = B_{12}\left(\frac{1}{2\pi} - \frac{4}{\pi^3}\right) + B_{15}.8/3\pi$$

$$\bar{B}_8 = 2B_{13}/\pi + B_{14}\left(\frac{1}{2\pi} - \frac{4}{\pi^3}\right)$$

$$\bar{C}_1 = C_1\pi$$

$$\bar{C}_2 = 3\ C_2/2 + 8C_7\pi^4$$

$$\bar{C}_3 = 3\ C_3/8\pi$$

$$\bar{C}_4 = 8\ C_4/3\pi$$

$$\bar{C}_5 = -\ 2\pi^2 C_5 + 3C_8\pi^2$$

$$\bar{C}_6 = -\ 3C_6/4 - 3C_{10}/2 - 2\pi^2 C_{13} - 4\pi^2 C_{14}\left(\frac{1}{24} - \frac{7}{16\pi^2}\right)$$

$$\bar{C}_7 = -\ 3C_9/4 + 3C_{11}/2$$

$$\bar{C}_8 = 3\ C_{12}/2$$

$$\bar{C}_9 = C_{15}$$

(b) <u>Hinged Supports</u>

Note: Only the following constants are changed.

$$\bar{A}_7 = -\ 4\ A_{12}/3$$

$$\bar{A}_8 = 8\ A_{13}/9\pi^2$$

$$\bar{B}_7 = B_{12}\left(\frac{1}{24} - \frac{1}{4\pi^2}\right) + B_{15}/2$$

$$\bar{C}_1 = C_1$$

$$\bar{C}_2 = C_2/2 + C_7 . \pi^4/2$$

$$\bar{C}_3 = 8\ C_3/9\pi^2$$

$$\bar{C}_4 = C_4/2$$

$$\bar{C}_5 = \frac{\pi^2}{2}\ (\ - C_5 + C_8/2\ )$$

$$\bar{C}_6 = - C_6/4 + C_{10}/2 + C_{13}(\ - \pi^2\ )/2$$
$$- \pi^2 C_{14}\left(\frac{1}{24} + \frac{1}{4\pi^2}\right)$$

$$\bar{C}_7 = - C_9/4 + C_{11}/2$$

$$\bar{C}_8 = C_{12}/2$$

$$\bar{C}_9 = 2\ C_{15}/\pi$$

$$G_{11} = a_3$$

$$G_{12} = - 8a_1a_3/a_4$$

$$G_{13} = -8a_3a_5/a_2a_4^2$$

$$G_{14} = \nu\, a_3/a_2a_4$$

$$G_{21} = G_{14}/\nu$$

$$G_{22} = -8a_3a_5/a_4$$

$$G_{23} = \nu\, a_3$$

$$G_{24} = -8a_1a_3\ /a_4$$

$$G_{31} = (1-\nu)a_3/2a_2a_4$$

$$G_{32} = (1-\nu)a_3/2$$

$$G_{33} = 64a_1ta_3/a_4^2$$

$$G_{34} = -4a_1t(1+\nu)\,a_3/a_4$$

$$G_{35} = G_{34}/a_2a_4$$

$$G_{41} = 8a_1ta_3/a_4^2$$

$$G_{42} = 8a_1t(1+\nu)a_3/a_4^2$$

$$G_{43} = -a_2a_3$$

$$G_{44} = -\nu a_3/a_2a_4^2$$

$$G_{51} = -a_3/a_2a_4^2$$

$$G_{52} = 8a_1t(1+\nu)a_3/a_4^2$$

$$G_{53} = -\nu a_2a_3$$

$$G_{54} = \nu G_{52}$$

$$G_{61} = -8a_3a_5/a_2a_4^3$$

$$G_{62} = a_3(1-\nu)/a_4$$

$$G_{71} = -G_{62}$$

$$G_{72} = 8a_3a_2a_5/a_4$$

APPENDIX B

Note: Only the non-zero elements of the matrices are listed.

$$D_1 = -8\ a_1 t/a_2 a_4^3$$

$$D_2 = 1/a_2^2 a_4^3$$

$$D_3 = -(1 - \nu)\ 8a_1 a_2 t/a_4$$

$$D_4 = 8(\nu - 2)a_1 t a_2/a_4$$

$$D_5 = (2 - \nu)/a_4$$

$$D_6 = 128\ a_1^2 t^2/a_4^3$$

$$D_7 = -24\ a_1 t/a_2 a_4^3$$

$$D_8 = 64\ a_1^2 t^2 (2 + \nu)/a_4^3$$

$$D_9 = 1/a_2 a_4$$

$$D_{10} = -8\ a_1 t(1 - \nu)/a_4$$

$$D_{11} = \nu$$

$$D_{12} = -8\ a_1 \nu/a_4$$

$$D_{13} = 1/a_2 a_4^2$$

$$D_{14} = -8\ a_1 t(1+\nu)/a_4^2$$

$$D_{15} = a_2 \nu$$

$$D_{16} = -8\ a_1 t\ \nu/a_4^2$$

$$D_{17} = 6/a_2 a_3^2 a_4$$

$$D_{18} = 6/a_3^2$$

$$D_{19} = 768\ a_1^2 t/a_4^2(1-\nu)a_3^2$$

$$D_{20} = -48\ a_1 t(1+\nu)/a_4 a_3^2(1-\nu)$$

$$D_{21} = -48\ a_1 t(1+\nu)/a_2 a_3^2 a_4^2(1-\nu)$$

$$D_{22} = 8\ a_1/a_2 a_4^2$$

$$D_{23} = -64\ a_1 t/a_4^2$$

$$D_{24} = -12\ a_2 t/a_3^2$$

$$D_{25} = \pi(-8\ D_3/3 + 64\ D_4/9)$$

$$D_{26} = -8\pi D_5/3 - 4\pi^2(1/6\pi - 52/27\pi^3)D_6 - 16D_8/9\pi$$

$$D_{28} = 8\ D_2/3\pi$$

$$D_{29} = D_{24}/t$$

$$D_{30} = \text{Cos } 2\theta$$

$$D_{31} = \text{Sin } 2\theta$$

$$D_{32} = D_{29}D_{31}$$

$$D_{33} = D_{29}D_{30}$$

(a) Fixed Supports:

$$(E)_{13} = -2\pi^2(-2D_3 + 3D_4)$$

$$(E)_{14} = -4\pi^2 D_5 + D_6(-8\pi^2)\left(\frac{1}{24} - \frac{7}{16\pi^2}\right) - 3D_8/2$$

$$(E)_{21} = 8\ D_{11}/3$$

$$(E)_{22} = -D_{10}/2$$

$$(E)_{33} = -2\pi^2 D_{15}$$

$$(E)_{34} = 3(D_{16} - D_{14}/2)/2$$

$$(E)_{41} = -D_{20}/4$$

$$(E)_{42} = -4D_{18}/3$$

$$(E)_{43} = 3D_{19}/8\pi + \pi D_{23}/2$$

$$(E)_{44} = -\pi D_{12}$$

$$(\overline{E})_{13} = 3(D_1 - D_7/2)$$

$$(\overline{E})_{14} = 3D_2$$

$$(\overline{E})_{22} = D_9$$

$$(\overline{E})_{23} = 3D_{13}/2$$

$$(\overline{E})_{41} = D_{17}/2$$

$$(\overline{E})_{42} = 8D_{21}/9\pi^2$$

(b) Hinged Supports:

$$(E)_{13} = \pi^2(-D_3 + D_4/2)$$

$$(E)_{14} = -\pi^2(D_5 + D_6(\tfrac{1}{12} - \tfrac{1}{2}\pi^2)) - D_8/2$$

$$(E)_{21} = 8D_{11}/3$$

$$(E)_{22} = -D_{10}/2$$

$$(E)_{23} = D_{12}$$

$$(E)_{33} = -\pi^2 D_{15}$$

$$(E)_{34} = D_{16} - D_{14}/2$$

$$(E)_{41} = -D_{20}/4$$

$$(E)_{42} = -4D_{18}/3$$

$$(E)_{43} = 8(D_{19}/\pi^2 - D_{23})/9$$

$$(E)_{44} = -4D_{22}/3$$

$$(\bar{E})_{13} = 3(D_1 - D_7/2)/2$$

$$(\bar{E})_{14} = 3D_2/2$$

$$(\bar{E})_{22} = D_9$$

$$(\bar{E})_{33} = D_{13}$$

$$(\bar{E})_{41} = D_{17}/2$$

$$(\bar{E})_{42} = 8D_{21}/9\pi^2$$

(a) <u>Fixed Supports</u>:

$$(S)_{12} = 1$$

$$(S)_{13} = -2t$$

$$(S)_{21} = \pi D_{24} \cdot D_{11}$$

$$(S)_{22} = -8D_{24} \cdot D_{10}/9\pi$$

$$(S)_{23} = -2\pi^2(-2D_3 + 3D_4) + D_{24} \cdot D_{12} \cdot 3/2$$

$$(S)_{3i} = (E)_{4i}$$

$$(\bar{S})_{22} = D_{24} \cdot D_9 \cdot 8/3\pi$$

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$$(\bar{S})_{23} = (\bar{E})_{13}$$

$$(\bar{S})_{24} = (\bar{E})_{14}$$

$$(\bar{S})_{3i} = (\bar{E})_{4i}$$

(b) <u>Hinged Supports</u>:

$$(S)_{12} = 1$$

$$(S)_{13} = -t$$

$$(S)_{21} = 8D_{24} \quad /3$$

$$(S)_{22} = - D_{24}D_{10}/2$$

$$(S)_{23} = D_{24}.D_{12} + \pi^2(-2D_3 + 3D_4)$$

$$(S)_{3i} = E_{4i}$$

$$(\bar{S})_{22} = D_{24}\ D_9$$

$$(\bar{S})_{23} = (\bar{E})_{13}$$

$$(\bar{S})_{24} = (\bar{E})_{14}$$

$$(\bar{S})_{3i} = (\bar{E})_{4i}$$

(a) <u>Fixed Supports</u>:

$$(X_i)_{11} = 1$$

$$(X_i)_{22} = \mathrm{Sin}\theta$$

$$(X_i)_{23} = 2\mathrm{Cos}\theta$$

$$(X_i)_{32} = - \mathrm{Cos}\theta$$

$$(X_i)_{33} = 2\mathrm{Sin}\theta$$

$$(X_i)_{44} = 1.$$

$$(X_j)_{1i} = (X_i)_{1i}$$

$$(X_j)_{2i} = (X_i)_{2i}$$

$$(X_j)_{3i} = -(X_i)_{3i}$$

$$(X_j)_{4i} = -(X_i)_{4i}$$

$$(A_{1i})_{1i} = (E)_{3i} \text{ (fixed)}$$

$$(A_{2i})_{1i} = (E)_{3i} \text{ (fixed)}$$

$$(A_{1j})_{1i} = (A_{1i})_{1i}$$

$$(A_{2j})_{1i} = (A_{2i})_{1i}$$

$$(A_{1i})_{2i} = (E)_{4i} \text{ (fixed)}$$

$$(A_{2i})_{2i} = (E)_{4i} \text{ (fixed)}$$

$$(A_{1j})_{2i} = (A_{1i})_{2i}$$

$$(A_{2j})_{2i} = (A_{2i})_{2i}$$

$$(A_{1i})_{33} = D_{25}$$

$$(A_{1i})_{34} = D_{26}$$

$$(A_{2i})_{33} = D_{27}$$

$$(A_{2i})_{34} = D_{28}$$

$$(A_{1j})_{31} = 4\, D_{32}D_{11}/3$$

$$(A_{1j})_{32} = -\, D_{32}D_{16}/2$$

$$(A_{1j})_{33} = 8D_{32}D_{12}/3\pi + D_{30}D_{25}$$

$$(A_{1j})_{34} = D_{30}\, D_{26}$$

$$(A_{2j})_{32} = D_{32}\, D_{9}/2$$

$$(A_{2j})_{33} = D_{30}\, D_{27}$$

$$(A_{2j})_{34} = D_{30}\, D_{28}$$

$$(A_{1i})_{41} = \pi D_{29}D_{11}$$

$$(A_{1i})_{42} = -\, 8\, D_{29}D_{10}/9\pi$$

$$(A_{1i})_{43} = 3\, D_{29}D_{12}/2$$

$$(A_{2i})_{42} = 8\, D_{29}D_{9}/3\pi$$

$$(A_{1j})_{41} = -\, \pi\, D_{33}D_{11}$$

$$(A_{1j})_{42} = 8D_{33}D_{10}/9\pi$$

$$(A_{1j})_{43} = -3D_{33}D_{12}/2 + 2D_{31}\pi^2(-2D_3 + D_4)$$

$$(A_{1j})_{44} = 2D_{31}\pi^2(-2D_3 + 3D_4)$$

$$(A_{2j})_{42} = -8D_{33}D_9/3\pi$$

$$(A_{2j})_{43} = 3D_{31}(D_1 - D_7/2)$$

$$(A_{2j})_{44} = 3D_{31}D_2$$

(b) Hinged Supports:

$$(X_i)_{11} = 1$$

$$(X_i)_{22} = \sin\theta$$

$$(X_i)_{23} = \cos\theta$$

$$(X_i)_{32} = -\cos\theta$$

$$(X_i)_{33} = \sin\theta$$

$$(X_i)_{44} = 1$$

$$(X_j)_{1i} = (X_i)_{1i}$$

$$(X_j)_{2i} = (X_i)_{1i}$$

$$(X_j)_{3i} = -(X_i)_{3i}$$

$$(X_j)_{4i} = -(X_i)_{4i}$$

$$(A_{1i})_{1i} = (E)_{3i} \text{ (hinged)}$$

$$(A_{2i})_{1i} = (\overline{E})_{3i} \text{ (hinged)}$$

$$(A_{1j})_{1i} = (A_{1i})_{1i}$$

$$(A_{2j})_{1i} = (A_{2i})_{1i}$$

$$(A_{1i})_{2i} = E_{4i} \text{ (hinged)}$$

$$(A_{2i})_{2i} = \overline{E}_{4i} \text{ (hinged)}$$

$$(A_{1j})_{2i} = (A_{1i})_{2i}$$

$$(A_{2j})_{2i} = (A_{2i})_{2i}$$

$$(A_{1i})_{33} = \pi^2(-D_3 + D_4/2)$$

$$(A_{1i})_{34} = -\pi^2(D_5 + D_6(\frac{1}{12} - \frac{1}{2\pi^2})) - D_8/2$$

$$(A_{2i})_{33} = D_1 - D_7/2$$

$$(A_{1j})_{43} = -D_{33}D_{12} + D_{31}\pi^2(-D_3 + D_4/2)$$

$$(A_{1j})_{44} = D_{31}(-\pi^2(D_5 + D_6(\frac{1}{12} - \frac{1}{2\pi^2})) - D_8/2)$$

$$(A_{2j})_{42} = -D_{33}D_9$$

$$(A_{2j})_{43} = D_{31}(D_1 - D_7/2)$$

$$(A_{2j})_{44} = D_{31}D_2$$

$$(A_{2i})_{34} = D_2$$

$$(A_{1j})_{31} = 8\ D_{32}D_9/3$$

$$(A_{1j})_{32} = -\ D_{32}D_{16}/2$$

$$(A_{1j})_{33} = D_{32}D_{12} + D_{30}\pi^2(-\ D_3 + D_4/2)$$

$$(A_{1j})_{34} = D_{30}(\pi^2\ (D_5 + D_6(\frac{1}{12} - \frac{1}{2\pi^2})) + D_8/2)$$

$$(A_{2j})_{32} = D_{32}D_9$$

$$(A_{2j})_{33} = D_{30}(D_1 - D_7/2)$$

$$(A_{2j})_{34} = D_{30}D_2$$

$$(A_{1i})_{41} = 8\ D_{29}D_{11}/3$$

$$(A_{1i})_{42} = -\ D_{29}D_{10}/2$$

$$(A_{1i})_{43} = D_{29}\ D_{12}$$

$$(A_{2i})_{42} = D_{29}\ D_9$$

$$(A_{1j})_{41} = -\ 8\ D_{33}D_{11}/3$$

$$(A_{1j})_{42} = D_{33}D_{10}/2$$